2-5 The Calculus of Scalar and Vector Fields (pp.33-55)

Q:

A:

1. 4.

2. 5.

3. 6.

A. The *Integration* of Scalar and Vector Fields

1. *The Line Integral*

$$\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Q1:

A1:
- HO: Differential Displacement Vectors
- HO: The Differential Displacement Vectors for Coordinate Systems

Q2:

A2: HO: The Line Ingtegral

Q3:

A3:
- HO: The Contour C
- HO: Line Integrals with Complex Contours

Q4:

A4: HO: Steps for Analyzing Line Integrals

Example: The Line Integral

2. *The Surface Integral*

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

Q1:

A1:

HO: Differential Surface Vectors

HO: The Differential Surface Vectors for Coordinate Systems

Q2:

A2: HO: The Surface Integral

Q3:

A3:
- HO: The Surface S
- HO: Integrals with Complex Surfaces

Q4:

A4:
- HO: Steps for Analyzing Surface Integrals
- Example: The Surface Integral

3. *The Volume Integral*

$$\iiint_V g(\bar{r})\, dv$$

Q1:

A1:

HO: The Differential Volume Element

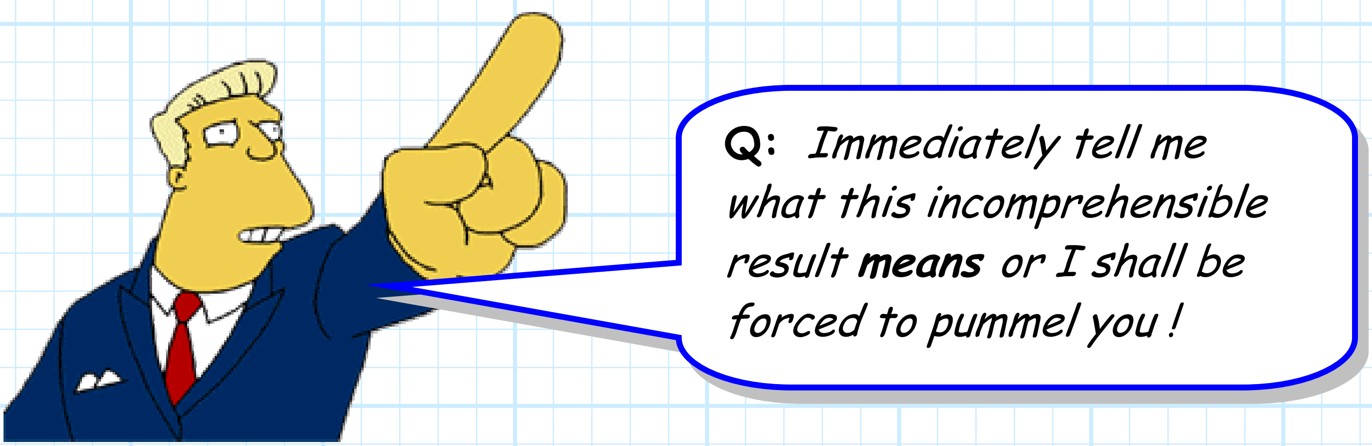
HO: The Volume V

Example: The Volume Integral

Differential Displacement Vectors

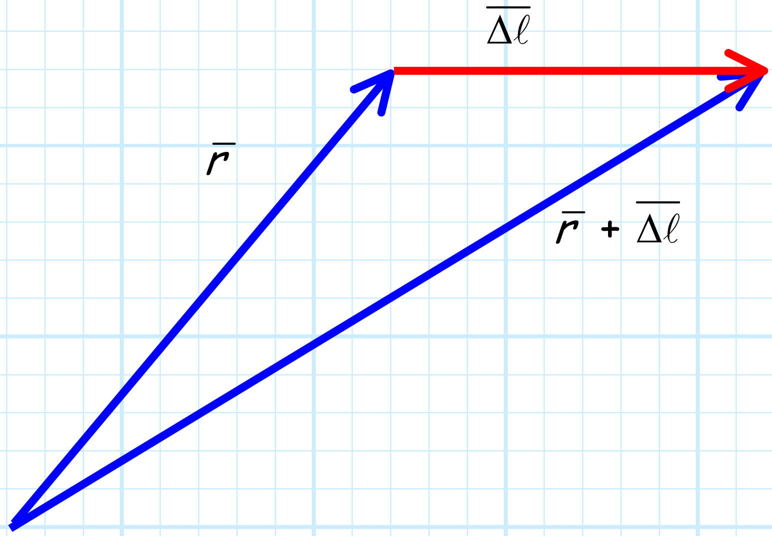
The derivative of a position vector $\bar{r}$, with respect to coordinate value ℓ (where $\ell \in \{x, y, z, \rho, \phi, r, \theta\}$) is expressed as:

$$
\begin{aligned}
\frac{d\bar{r}}{d\ell} &= \frac{d}{d\ell}\left(x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z\right) \\
&= \frac{d\left(x\,\hat{a}_x\right)}{d\ell} + \frac{d\left(y\,\hat{a}_y\right)}{d\ell} + \frac{d\left(z\,\hat{a}_z\right)}{d\ell} \\
&= \left(\frac{dx}{d\ell}\right)\hat{a}_x + \left(\frac{dy}{d\ell}\right)\hat{a}_y + \left(\frac{dz}{d\ell}\right)\hat{a}_z
\end{aligned}
$$



A: The vector above describes the **change** in **position vector** $\bar{r}$ due to a change in coordinate variable ℓ. This change in position vector is itself a vector, with both a **magnitude** and **direction**.

For example, if a point moves such that its coordinate ℓ changes from ℓ to $\ell + \Delta\ell$, then the position vector that describes that point changes from $\bar{r}$ to $\bar{r} + \overline{\Delta\ell}$.



In other words, this small vector $\overline{\Delta\ell}$ is simply a **directed distance** between the point at coordinate ℓ and its new location at coordinate $\ell + \Delta\ell$!

This directed distance $\overline{\Delta\ell}$ is related to the position vector derivative as:

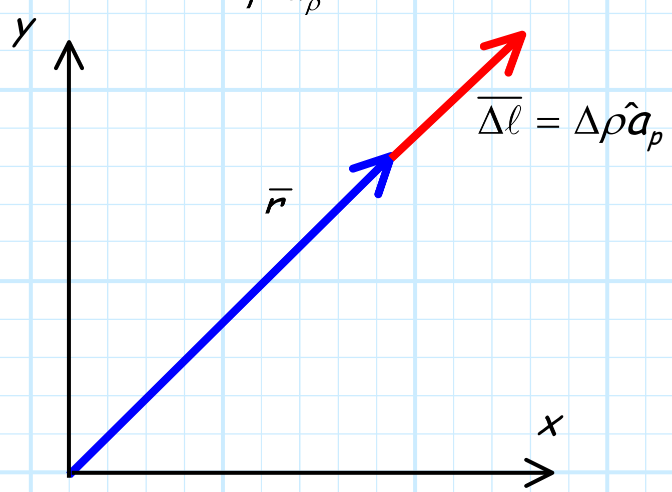
$$\begin{aligned}\overline{\Delta\ell} &= \Delta\ell \ \frac{d\,\bar{r}}{d\ell} \\ &= \Delta\ell\left(\frac{dx}{d\ell}\right)\hat{a}_x + \Delta\ell\left(\frac{dy}{d\ell}\right)\hat{a}_y + \Delta\ell\left(\frac{dz}{d\ell}\right)\hat{a}_z\end{aligned}$$

As an **example**, consider the case when $\ell = \rho$. Since $x = \rho\cos\phi$ and $y = \rho\sin\phi$ we find that:

$$\begin{aligned}\frac{d\overline{r}}{d\rho} &= \frac{dx}{d\rho}\hat{a}_x + \frac{dy}{d\rho}\hat{a}_y + \frac{dz}{d\rho}\hat{a}_z \\ &= \frac{d(\rho\cos\phi)}{d\rho}\hat{a}_x + \frac{d(\rho\sin\phi)}{d\rho}\hat{a}_y + \frac{dz}{d\rho}\hat{a}_z \\ &= \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y \\ &= \hat{a}_\rho\end{aligned}$$

A change in position from coordinates ρ, ϕ, z to $\rho + \Delta\rho, \phi, z$ results in a change in the position vector from $\overline{r}$ to $\overline{r}$ + $\overline{\Delta\ell}$. The vector $\overline{\Delta\ell}$ is a directed distance extending from point ρ, ϕ, z to point $\rho + \Delta\rho, \phi, z$, and is equal to:

$$\begin{aligned}\overline{\Delta\ell} &= \Delta\rho\,\frac{d\overline{r}}{d\rho} \\ &= \Delta\rho\,\cos\phi\,\hat{a}_x + \Delta\rho\,\sin\phi\,\hat{a}_y \\ &= \Delta\rho\,\hat{a}_\rho\end{aligned}$$



If $\Delta\ell$ is really small (i.e., as it approaches zero) we can define something called a **differential displacement vector** $\overline{d\ell}$:

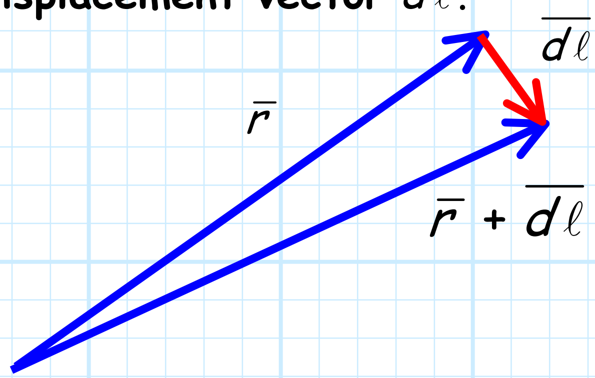
$$\begin{aligned}\overline{d\ell} &\doteq \lim_{\Delta\ell \to 0} \overline{\Delta\ell} \\ &= \lim_{\Delta\ell \to 0} \left(\frac{d\,\overline{r}}{d\,\ell}\right)\Delta\ell \\ &= \left(\frac{d\,\overline{r}}{d\,\ell}\right) d\,\ell\end{aligned}$$

For example:

$$\overline{d\rho} = \frac{d\,\overline{\mathrm{r}}}{d\,\rho}\,d\,\rho = \hat{a}_\rho\,d\,\rho$$

Essentially, the differential line vector $\overline{d\ell}$ is the **tiny directed distance** formed when a point changes it location by some tiny amount, resulting in a change of one coordinate value ℓ by an equally tiny (i.e., differential) amount $d\,\ell$.

The **directed distance** between the original location (at coordinate value ℓ) and its new location (at coordinate value $\ell + d\ell$) is the **differential displacement vector** $\overline{d\ell}$.



We will use the differential line vector when evaluating a **line integral**.

The Differential Displacement Vector for Coordinate Systems

Let's determine the **differential displacement vectors** for each coordinate of the Cartesian, cylindrical and spherical coordinate systems!

Cartesian

This is easy!

$$\overline{dx} = \frac{d\bar{r}}{dx}\, dx = \left[\left(\frac{d\,x}{dx}\right)\hat{a}_x + \left(\frac{d\,y}{dx}\right)\hat{a}_y + \left(\frac{d\,z}{dx}\right)\hat{a}_z\right] dx$$
$$= \hat{a}_x\, dx$$

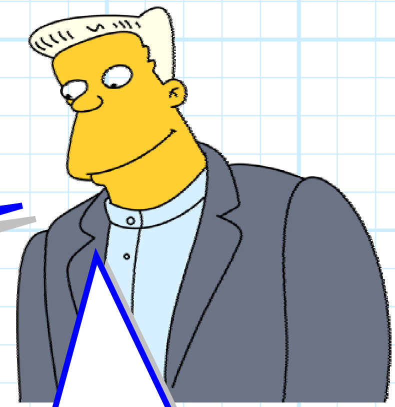
$$\overline{dy} = \frac{d\bar{r}}{dy}\, dy = \left[\left(\frac{d\,x}{dy}\right)\hat{a}_x + \left(\frac{d\,y}{dy}\right)\hat{a}_y + \left(\frac{d\,z}{dy}\right)\hat{a}_z\right] dy$$
$$= \hat{a}_y\, dy$$

$$\overline{dz} = \frac{d\bar{r}}{dz}\, dz = \left[\left(\frac{d\,x}{dz}\right)\hat{a}_x + \left(\frac{d\,y}{dz}\right)\hat{a}_y + \left(\frac{d\,z}{dz}\right)\hat{a}_z\right] dz$$
$$= \hat{a}_z\, dz$$

Cylindrical

Likewise, recall from the last handout that:

$$\overline{d\rho} = \hat{a}_\rho \, d\rho$$



*Maria, look! I'm starting to see a **trend!***

$$\overline{dx} = \frac{d\overline{r}}{dx} dx = \hat{a}_x \, dx$$

$$\overline{dy} = \frac{d\overline{r}}{dy} dy = \hat{a}_y \, dy$$

$$\overline{dz} = \frac{d\overline{r}}{dz} dz = \hat{a}_z \, dz$$

$$\overline{d\rho} = \frac{d\overline{r}}{d\rho} d\rho = \hat{a}_\rho \, d\rho$$

***Q:** It seems **very** apparent that:*

$$\overline{d\ell} = \hat{a}_\ell \, d\ell$$

*for **all** coordinates ℓ; right ?*

A: NO!! Do not make this mistake! For example, consider $\overline{d\phi}$:

***Q:** No!! $\overline{d\phi} = \hat{a}_\phi \, \rho \, d\phi$?!?*

How did the coordinate ρ get in there?

$$\begin{aligned}
\overline{d\phi} &= \frac{d\overline{r}}{d\phi} d\phi \\
&= \left(\frac{dx}{d\phi}\hat{a}_x + \frac{dy}{d\phi}\hat{a}_y + \frac{dz}{d\phi}\hat{a}_z \right) d\phi \\
&= \left(\frac{d\,\rho\cos\phi}{d\phi}\hat{a}_x + \frac{d\,\rho\sin\phi}{d\phi}\hat{a}_y + \frac{dz}{d\phi}\hat{a}_z \right) d\phi \\
&= \left(-\rho\sin\phi\,\hat{a}_x + \rho\cos\phi\,\hat{a}_y \right) d\phi \\
&= \left(-\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y \right) \rho \, d\phi = \hat{a}_\phi \, \rho \, d\phi
\end{aligned}$$

The scalar differential value $\rho \, d\phi$ **makes sense**! The differential displacement vector is a **directed distance**, thus the units of its magnitude must be **distance** (e.g., meters, feet). The differential value $d\phi$ has units of **radians**, but the differential value $\rho \, d\phi$ **does** have units of distance.

The differential displacement vectors for the **cylindrical** coordinate system is therefore:

$$\overline{d\rho} = \frac{d\overline{r}}{d\rho} d\rho = \hat{a}_\rho \, d\rho$$

$$\overline{d\phi} = \frac{d\overline{r}}{d\phi} d\phi = \hat{a}_\phi \, \rho \, d\phi$$

$$\overline{dz} = \frac{d\overline{r}}{dz} dz = \hat{a}_z \, dz$$

Likewise, for the **spherical** coordinate system, we find that:

$$\overline{dr} = \frac{d\overline{r}}{dr} dr = \hat{a}_r \, dr$$

$$\overline{d\theta} = \frac{d\overline{r}}{d\theta} d\theta = \hat{a}_\theta \, r \, d\theta$$

$$\overline{d\phi} = \frac{d\overline{r}}{d\phi} d\phi = \hat{a}_\phi \, r \sin\theta \, d\phi$$

The Line Integral

This integral is alternatively known as the **contour integral**. The reason is that the line integral involves integrating the projection of a vector field onto a specified **contour** C, e.g.,

$$\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Some important things to note:

* The integrand is a **scalar** function.

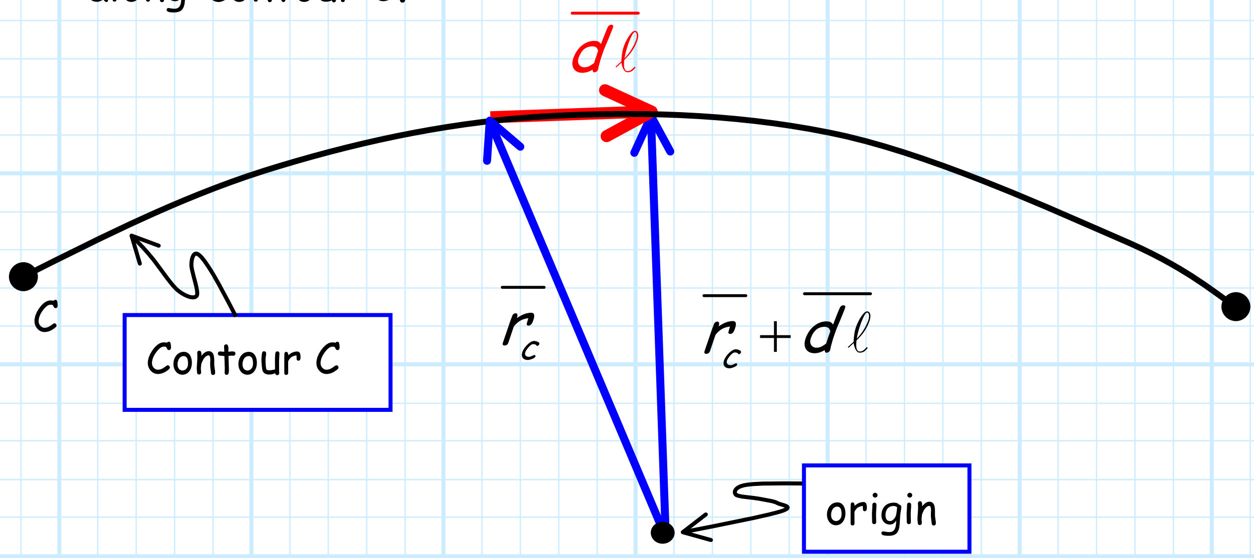
* The integration is over **one** dimension.

* The **contour** C is a line or curve through three-dimensional space.

* The position vector $\bar{r}_c$ denotes only those points that lie on contour C. Therefore, the value of this integral **only** depends on the value of vector field $\mathbf{A}(\bar{r})$ at the points along this contour.

Q: *What is the differential vector $\overline{d\ell}$, and how does it relate to contour C?*

A: The differential vector $\overline{d\ell}$ is the tiny **directed distance** formed when a point moves a small distance along contour C.



As a result, the differential line vector $\overline{d\ell}$ is **always tangential** to every point of the contour. In other words, the direction of $\overline{d\ell}$ always points "down" the contour.

Q: *So what does the scalar integrand $\mathbf{A}(\overline{r}_c)\cdot\overline{d\ell}$ mean? What is it that we are actually integrating?*

A: Essentially, the line integral integrates (i.e., "adds up") the values of a **scalar component** of vector field $\mathbf{A}(\overline{r})$ at **each and every point** along contour C. This scalar component of vector field $\mathbf{A}(\overline{r})$ is the projection of $\mathbf{A}(\overline{r}_c)$ onto the direction of the contour C.

First, I must point out that the notation $\mathbf{A}(\bar{r}_c)$ is **non-standard**. Typically, the vector field in the line integral is denoted simply as $\mathbf{A}(\bar{r})$. I use the notation $\mathbf{A}(\bar{r}_c)$ to emphasize that we are integrating the values of the vector field $\mathbf{A}(\bar{r})$ **only** at point that lie on contour C, and the points that lie on contour C are denoted as position vector $\bar{r}_c$.

In other words, the values of vector field $\mathbf{A}(\bar{r})$ at points that do not lie on the contour (which is just about all of them!) have no effect on the integration. The integral **only** depends on the value of the vector field as we move along contour C—we denote these values as $\mathbf{A}(\bar{r}_c)$.

Moreover, the line integral depends on **only one component** of $\mathbf{A}(\bar{r}_c)$!

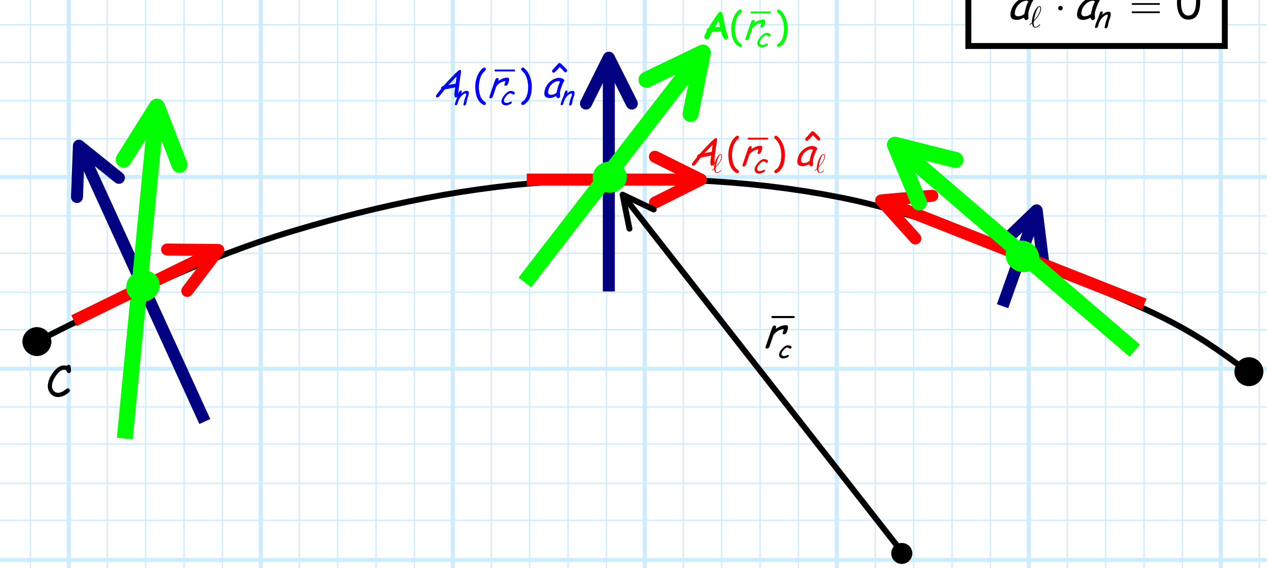
Q: *On just what component of* $\mathbf{A}(\bar{r}_c)$ ***does*** *the integral depend?*

A: Look at the integrand $\mathbf{A}(\bar{r}_c)\cdot\overline{d\ell}$ --we see it involves the **dot** product! Thus, we find that the scalar integrand is simply the **scalar projection** of $\mathbf{A}(\bar{r}_c)$ onto the differential vector $\overline{d\ell}$. As a result, the integrand depends **only** the component of $\mathbf{A}(\bar{r}_c)$ that lies in the direction of $\overline{d\ell}$--and $\overline{d\ell}$ **always** points in the direction of the contour C!

To help see this, first note that $\mathbf{A}(\bar{r}_c)$, the value of the vector field along the contour, can be written in terms of a vector component **tangential** to the contour (i.e, $A_\ell(\bar{r}_c)\,\hat{a}_\ell$), and a vector component that is **normal** (i.e., orthogonal) to the contour (i.e, $A_n(\bar{r}_c)\,\hat{a}_n$):

$$\mathbf{A}(\bar{r}_c) = A_\ell(\bar{r}_c)\,\hat{a}_\ell + A_n(\bar{r}_c)\,\hat{a}_n$$

$$\hat{a}_\ell \cdot \hat{a}_n = 0$$



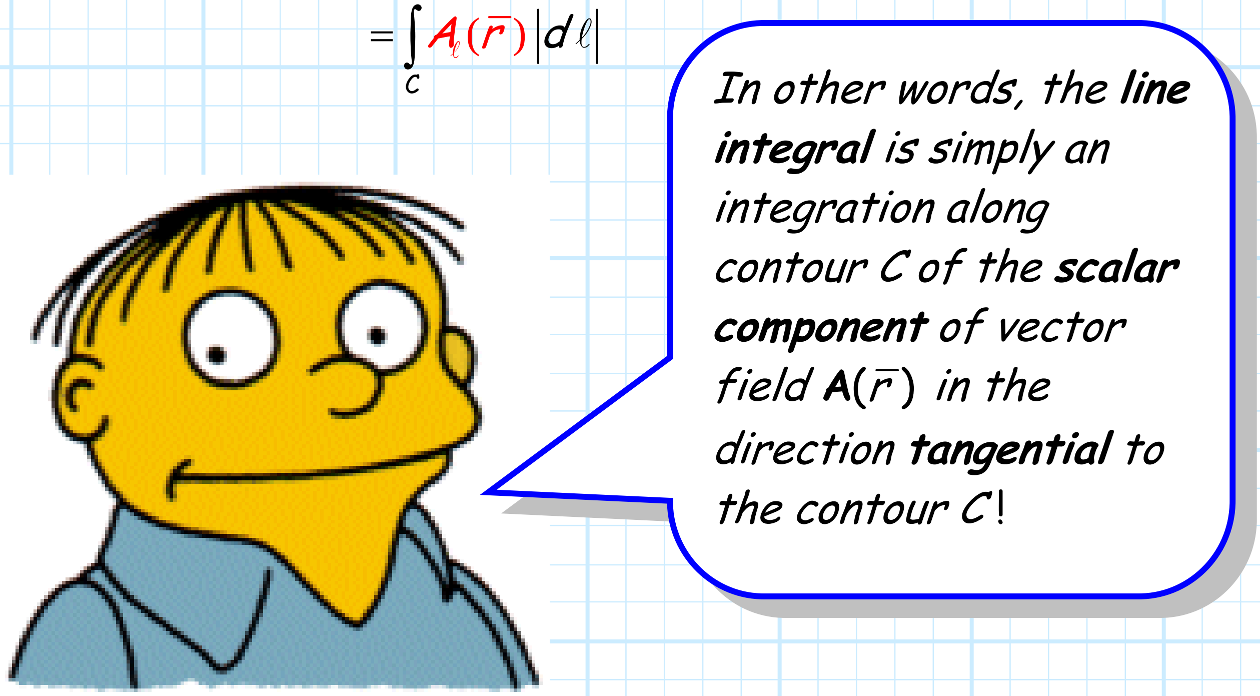
We likewise note that the differential line vector $\overline{d\ell}$, like **any** and all vectors, can be written in terms of its magnitude ($|d\ell|$) and direction ($\hat{a}_\ell$) as:

$$\overline{d\ell} = \hat{a}_\ell\,|d\ell|$$

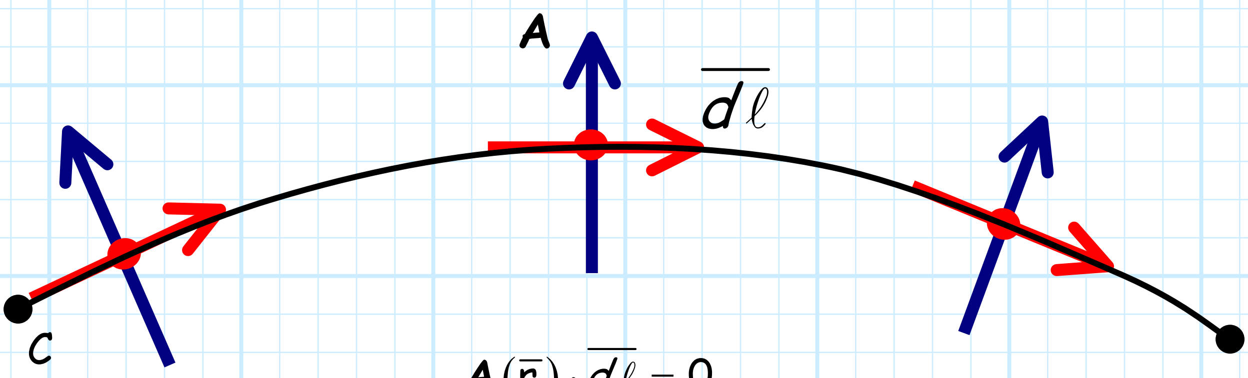
For example, for $\overline{d\phi} = \rho\, d\phi\, \hat{a}_\phi$, we can say $|d\ell| = \rho\, d\phi$ and $\hat{a}_\ell = \hat{a}_\phi$.

As a result we can write:

$$\begin{aligned}\int_C \mathbf{A}(\bar{\mathbf{r}}_c)\cdot\overline{d\ell} &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\overline{d\ell}\\ &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\right]\cdot\ \hat{a}_\ell\,|d\ell|\\ &= \int_C \left[A_\ell(\bar{r})\,\hat{a}_\ell\cdot\hat{a}_\ell + A_n(\bar{r})\,\hat{a}_n\cdot\hat{a}_\ell\right]|d\ell|\\ &= \int_C A_\ell(\bar{r})\,|d\ell|\end{aligned}$$



Note if vector field $\mathbf{A}(\bar{\mathbf{r}})$ is **orthogonal** to the contour at every point, then the resulting line integral will be **zero**.



$$\mathbf{A}(\bar{\mathbf{r}})\cdot\overline{d\ell} = 0$$

Although C represents **any** contour, no matter how **complex** or **convoluted**, we will study only **basic** contours. In other words, $\overline{d\ell}$ will correspond to one of the differential line vectors we have **previously** determined for Cartesian, cylindrical, and spherical coordinate systems.

The Contour C

In this class, we will limit ourselves to studying only those contours that are formed when we change the location of a point by varying **just one** coordinate parameter. In other words, the other two coordinate parameters will remain **fixed**.

Mathematically, therefore, a **contour** is described by:

2 equalities (e.g., $x=2$, $y=-4$; $r=3$, $\phi=\pi/4$)

AND

1 inequality (e.g., $-1 < z < 5$; $0 < \theta < \pi/2$)

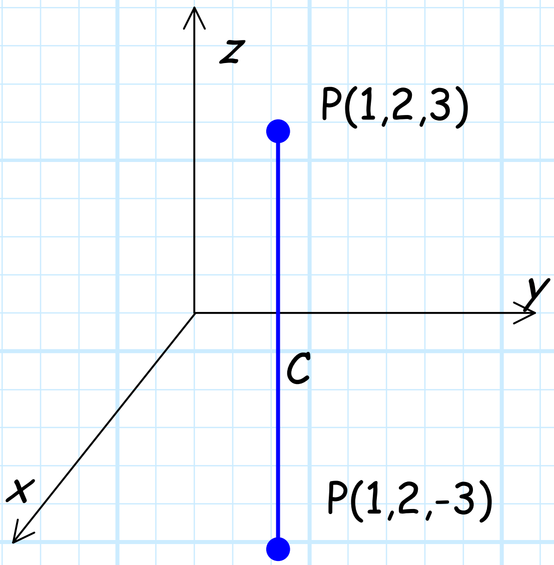
Likewise, we will need to explicitly determine the differential displacement vector $\overline{d\ell}$ for each contour.

Recall we have studied **seven** coordinate parameters ($x, y, z, \rho, \phi, r, \theta$). As a result, we can form **seven** different contours C!

Cartesian Contours

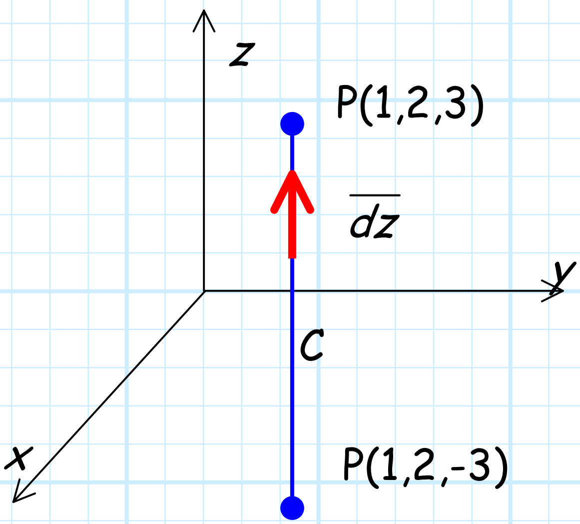
Say we move a point from P($x=1$, $y=2$, $z=-3$) to P($x=1$, $y=2$, $z=3$) by changing **only** the coordinate variable z from $z=-3$ to $z=3$. In other words, the coordinate values x and y remain **constant** at $x=1$ and $y=2$.

We form a contour that is a **line segment**, **parallel** to the z-axis!



Note that **every** point along this segment has coordinate values x =1 and y =2. As we move along the contour, the **only** coordinate value that changes is z.

Therefore, the **differential** directed distance associated with a change in position from z to $z+dz$, is $\overline{d\ell} = \overline{dz} = \hat{a}_z dz$.



Similarly, a line segment parallel to the x-axis (or y-axis) can be formed by changing coordinate parameter x (or y), with a resulting differential displacement vector of $\overline{d\ell} = \overline{dx} = \hat{a}_x dx$ (or $\overline{d\ell} = \overline{dy} = \hat{a}_y dy$).

The three Cartesian contours are therefore:

1. Line segment parallel to the z-axis

$$x = c_x \qquad y = c_y \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{d\ell} = \hat{a}_z \, dz$$

2. Line segment parallel to the y-axis

$$x = c_x \qquad c_{y1} \leq y \leq c_{y2} \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_y \, dy$$

3. Line segment parallel to the x-axis

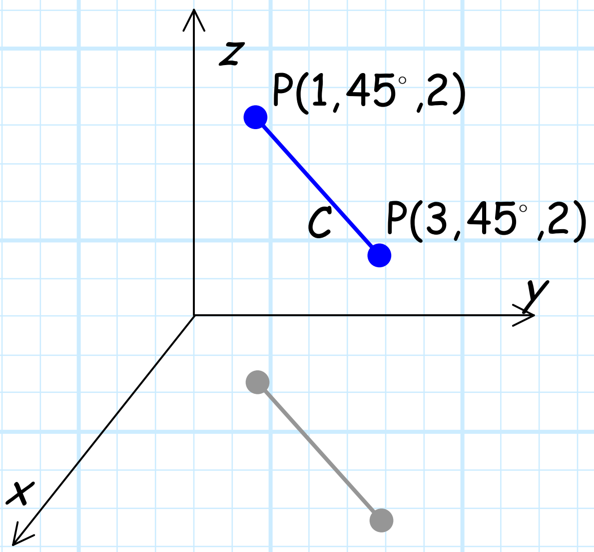
$$c_{x1} \leq x \leq c_{x2} \qquad y = c_y \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_x \, dx$$

Cylindrical Contours

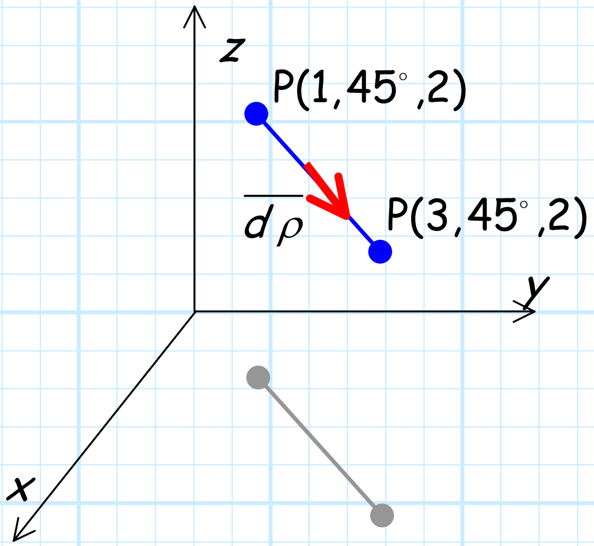
Say we move a point from P(ρ=1, $\phi = 45^\circ$, z =2) to P(ρ=3, $\phi = 45^\circ$, z =2) by changing **only** the coordinate variable ρ from ρ =*1* to ρ=3. In other words, the coordinate values ϕ and z remain **constant** at $\phi = 45^\circ$ and z=2.

We form a contour that is a **line segment**, **parallel** to the *x-y* plane (i.e., perpendicular to the *z*-axis).



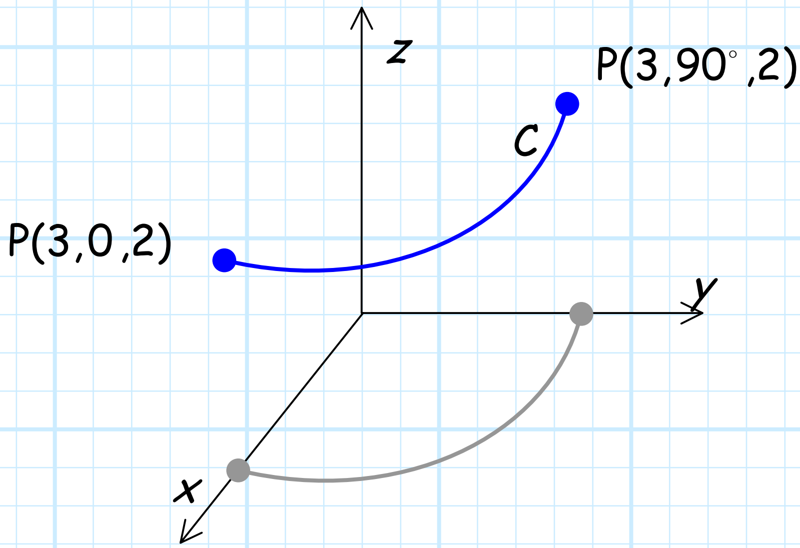
Note that **every** point along this segment has coordinate values $\phi = 45^\circ$ and z=2. As we move along the contour, the **only** coordinate value that changes is ρ.

Therefore, the **differential** directed distance associated with a change in position from ρ to $\rho + d\rho$, is $\overline{d\ell} = \overline{d\rho} = \hat{a}_\rho \, d\rho$.



Alternatively, say we move a point from P(ρ=3, $\phi = 0$, z=2) to P(ρ=3, $\phi = 90^\circ$, z=2) by changing **only** the coordinate variable ϕ from ϕ=0 to $\phi = 90^\circ$. In other words, the coordinate values ρ and z remain **constant** at ρ=3 and z=2.

We form a contour that is a **circular arc**, parallel to the *x-y* plane.

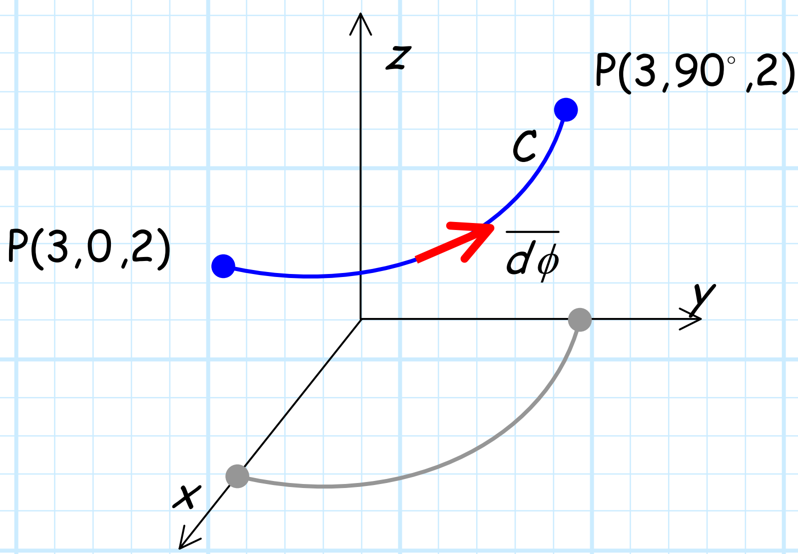


Note: if we move from $\phi = 0$ to $\phi = 360^\circ$, a complete **circle** is formed around the *z*-axis.

Every point along the arc has coordinate values $\rho = 3$ and $z = 2$. As we move along the contour, the **only** coordinate value that changes is ϕ.

Therefore, the **differential** directed distance associated with a change in position from ϕ to $\phi + d\phi$, is:

$$\overline{d\ell} = \overline{d\phi} = \hat{a}_\phi \; \rho d\phi .$$



Finally, changing coordinate *z* generates the **third** cylindrical contour—but we **already** did that in Cartesian coordinates! The result is **again** a line segment parallel to the z-axis.

The three cylindrical contours are therefore described as:

1. *Line segment parallel to the z- axis.*

$$\rho = c_\rho \qquad \phi = c_\phi \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{d\ell} = \hat{a}_z \, dz$$

2. *Circular arc parallel to the x-y plane.*

$$\rho = c_\rho \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_\phi \;\; \rho d\phi$$

3. *Line segment parallel to the x-y plane.*

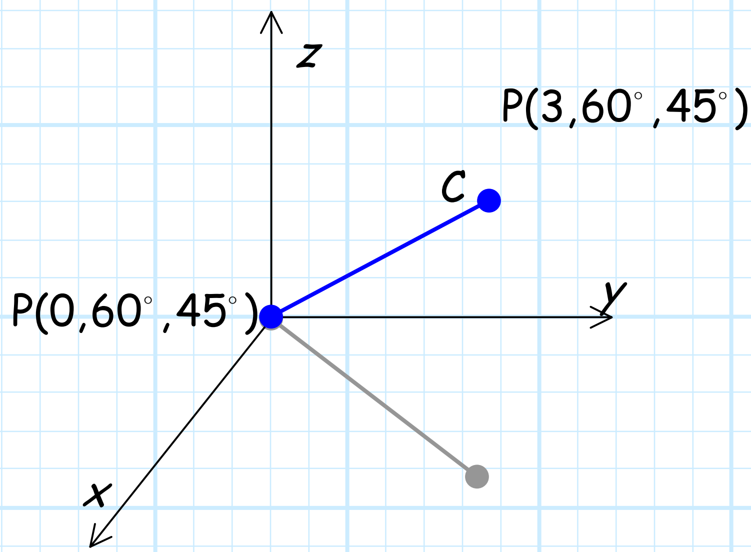
$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad \phi = c_\phi \qquad z = c_z$$

$$\overline{d\ell} = \hat{a}_\rho \;\; d\rho$$

Spherical Contours

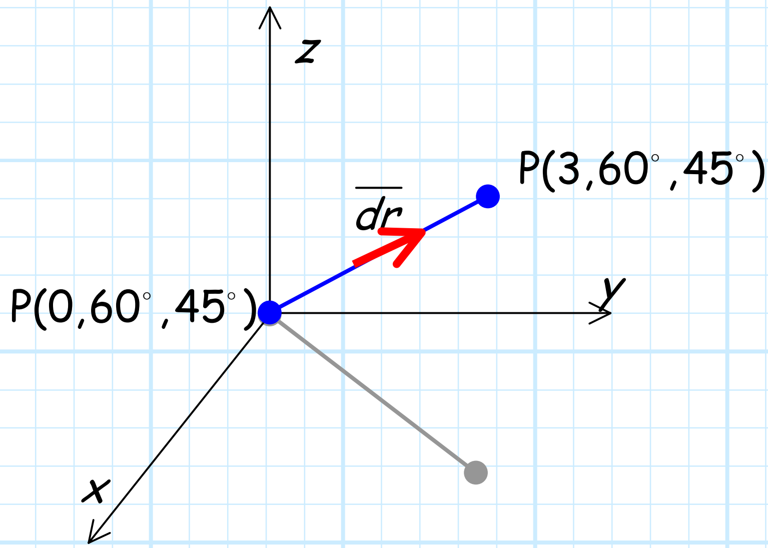
Say we move a point from P(r =0, $\theta = 60^\circ, \phi = 45^\circ$) to P($r$ =3, $\theta = 60^\circ, \phi = 45^\circ$) by changing **only** the coordinate variable r from r=0 to r =3. In other words, the coordinate values θ and ϕ remain **constant** at $\theta = 60^\circ$ and $\phi = 45^\circ$.

We form a contour that is a **line segment**, emerging from the **origin**.



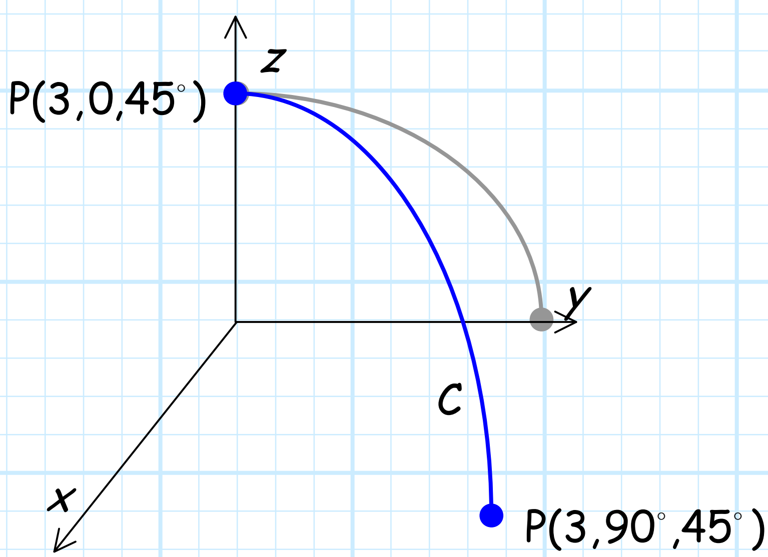
Every point along the line segment has coordinate values $\theta = 60^\circ$ and $\phi = 45^\circ$. As we move along the contour, the **only** coordinate value that changes is r.

Therefore, the **differential** directed distance associated with a change in position from r to $r + dr$, is $\overline{d\ell} = \overline{dr} = \hat{a}_r\, dr$.



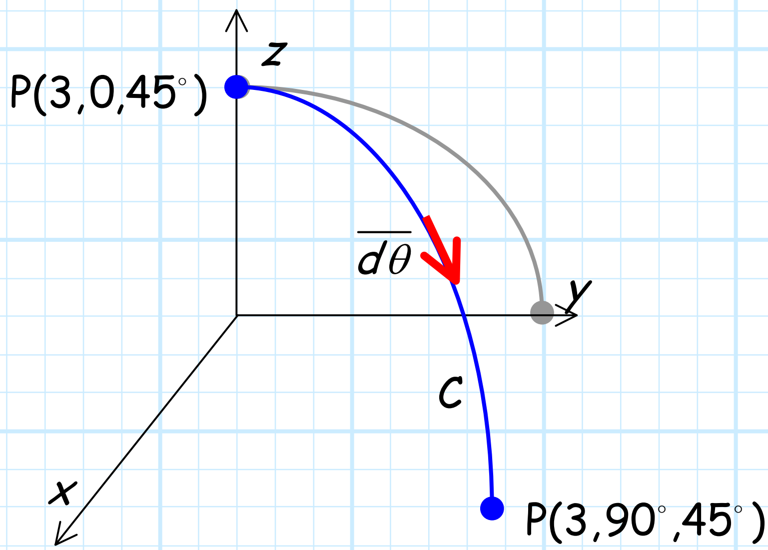
Alternatively, say we move a point from P(r =3, $\theta = 0$,$\phi = 45^\circ$) to P(r =3, $\theta = 90^\circ$,$\phi = 45^\circ$) by changing **only** the coordinate variable θ from $\theta = 0$ to $\theta = 90^\circ$. In other words, the coordinate values θ and ϕ remain **constant** at $\theta = 60^\circ$ and $\phi = 45^\circ$.

We form a **circular arc**, whose plane includes the z-axis.



Every point along the arc has coordinate values $r = 3$ and $\phi = 45^\circ$. As we move along the contour, the **only** coordinate value that changes is θ.

Therefore, the **differential** directed distance associated with a change in position from θ to $\theta + d\theta$, is $\overline{d\ell} = \overline{d\theta} = \hat{a}_\theta \, r d\theta$.



Finally, we could fix coordinates r and θ and vary coordinate ϕ only—but we **already** did this in cylindrical coordinates! We **again** find that a **circular arc** is generated, an arc that is parallel to the x-y plane.

The three spherical contours are therefore:

1. *A circular arc parallel to the x-y plane.*

$$r = c_r \qquad \theta = c_\theta \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{d\ell} = \hat{a}_\phi \;\; r \sin\theta \, d\phi$$

2. *A circular arc in a plane that includes the z-axis.*

$$r = c_r \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad \phi = c_\phi$$

$$\overline{d\ell} = \hat{a}_\theta \;\; r \, d\theta$$

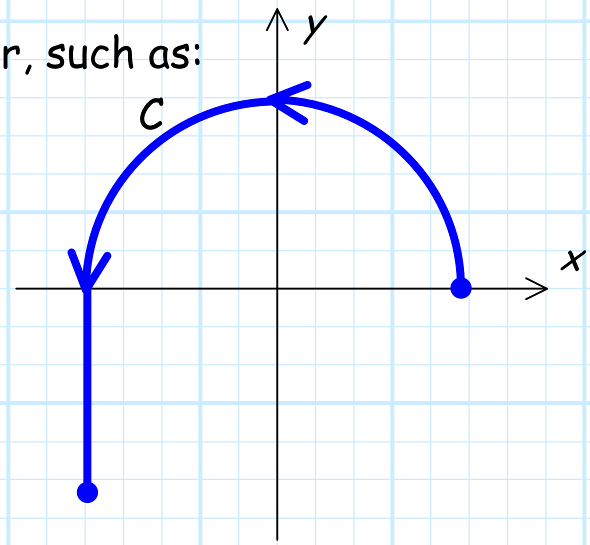
3. *A line segment directed toward the origin.*

$$c_{r1} \leq r \leq c_{r2} \qquad \theta = c_\theta \qquad \phi = c_\phi$$

$$\overline{d\ell} = \hat{a}_r \;\; dr$$

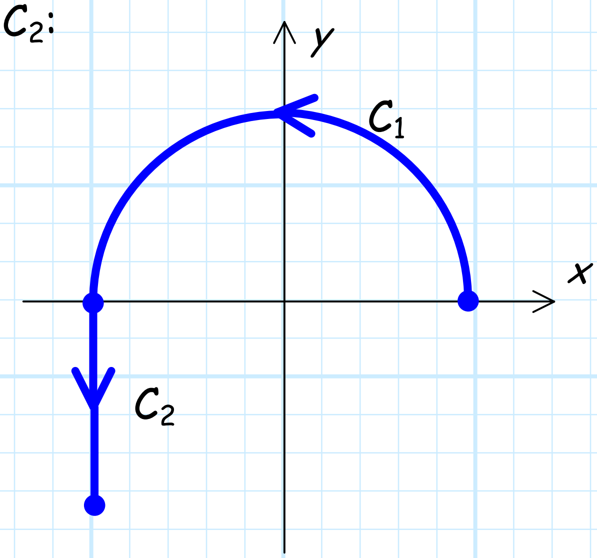
Line Integrals with Complex Contours

Consider a more complex contour, such as:



Q: What's this flim-flam?! **This** contour can **neither** be expressed in terms of **single** coordinate inequality, **nor** with **single** differential line vector!

A: True! But we can still **easily** evaluate a line integral over this contour C. The trick is to divide C into **two** contours, denoted as C_1 and C_2:



We can denote contour C as $C = C_1 + C_2$. It can be shown that:

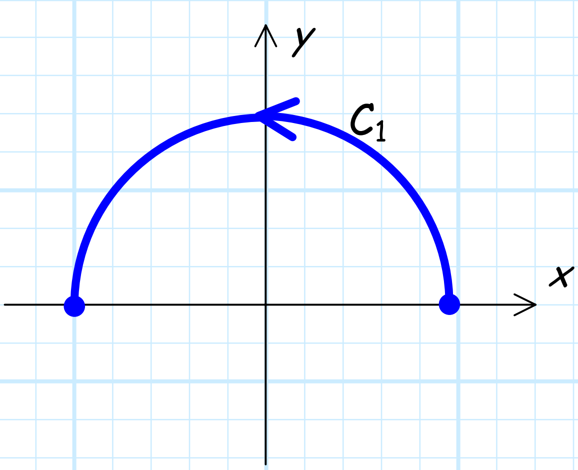
$$\int_C \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} = \int_{C_1} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} + \int_{C_2} \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell}$$

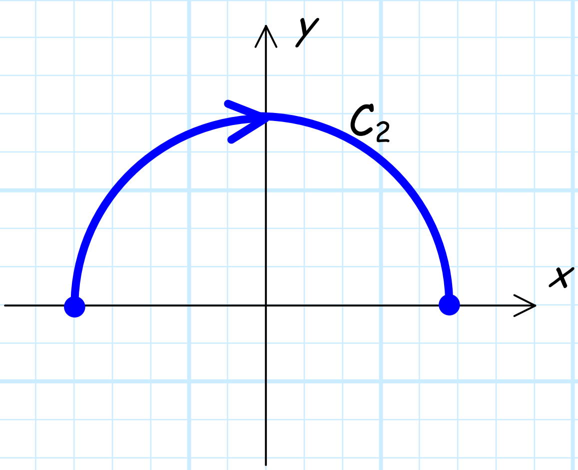
Note for the example given, we can evaluate the integral over both contour C_1 and contour C_2. The first is a **circular arc** around the z-axis, and the second is a **line segment** parallel to the y-axis.

Q: *Does the direction of the contour matter?*

A: **YES!** Every contour has a **starting** point and an **end** point. Integrating along the contour in the **opposite** direction will result in an **incorrect** answer!

For example, consider the two contours below:





In this case, the two contours are identical, with the **exception** of **direction**. In other words the beginning point of one is the end point of the other, and vice versa.

For this example, we would relate the two contours by saying:

$$C_1 = -C_2 \quad \text{and/or} \quad C_2 = -C_1$$

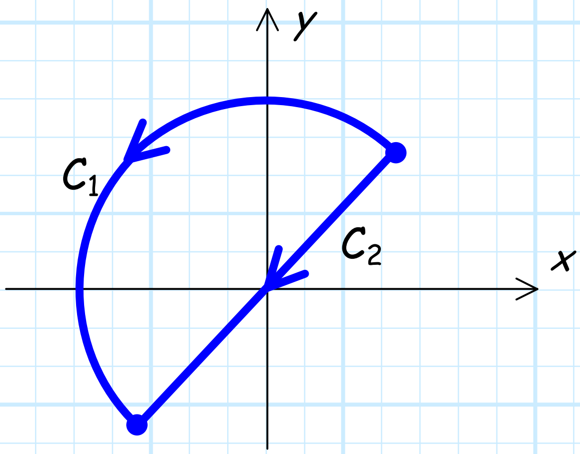
Just like vectors, the **negative** of a contour is an otherwise identical contour with opposite direction. We find that:

$$\int_{-C} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} = -\int_{C} \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

Q: *Does the* ***shape*** *of the contour* ***really*** *matter, or does the result of line integration only depend on the starting and end points ??*

A: Generally speaking, the shape of the contour **does** matter. Not only does the line integral depend on where we start and where we finish, it also depend on the path we take to get there!

For example, consider these two contours:



Generally speaking, we find that:

$$\int_{C_1} \mathbf{A}(\bar{r}_c)\cdot \overline{d\ell} \neq \int_{C_2} \mathbf{A}(\bar{r}_c)\cdot \overline{d\ell}$$

An **exception** to this is a **special** category of vector fields called **conservative** fields. For conservative fields, the contour path does **not** matter—the beginning and end points of the contour are **all** that are required to evaluate a line integral !

Steps for Analyzing Line Integrals

You wish to evaluate an integral of the form:

$$\int_C \mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$$

To successfully accomplish this, simply follow these steps:

Step 1: Determine the 2 equalities, 1 inequality, and $\overline{d\ell}$ for the **contour** C.

Step 2: Evaluate the **dot product** $\mathbf{A}(\bar{r}) \cdot \overline{d\ell}$.

Step 3: Transform all coordinates of the resulting **scalar** field to the **same** system as C.

Step 4: Evaluate the scalar field using the **two** coordinate **equalities** that describe contour C.

Step 5: Determine the **limits of integration** from the **inequality** that describes contour C *(be careful of order!).*

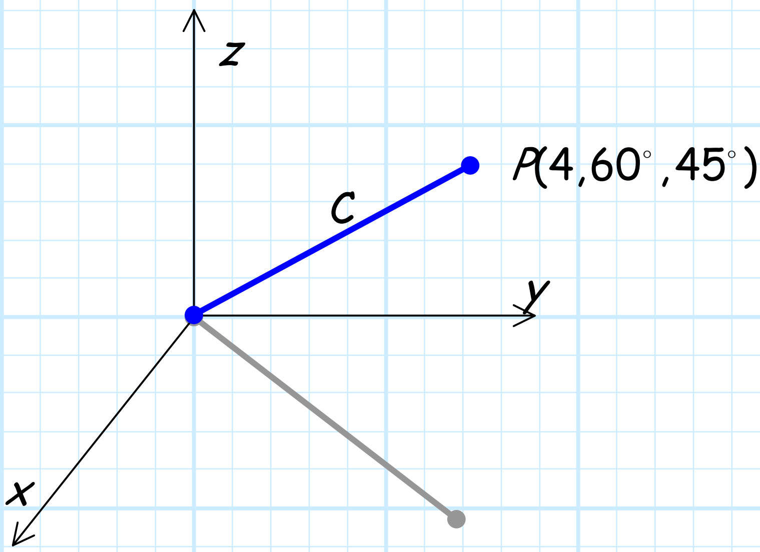
Step 6: Integrate the remaining function of **one** coordinate variable.

Example: The Line Integral

Consider the vector field:

$$\mathbf{A}(\bar{r}_c) = z\ \hat{a}_x - x\ \hat{a}_y$$

Integrate this vector field over **contour** C, a straight line that begins at the **origin** and ends at point $P(r = 4, \theta=60^\circ, \phi=45^\circ)$.



Step 1: Determine the **two equalities**, **one inequality**, and proper $\overline{d\ell}$ for the **contour** C.

This contour is formed as the coordinate r changes from r=0 to r=4, where $\theta = 60^\circ$ and $\phi = 45^\circ$ for all points. The two equalities and one inequality that define this contour are thus:

$$0 \leq r \leq 4 \qquad \theta = 60^\circ \qquad \phi = 45^\circ$$

and the **differential** displacement vector for this contour is therefore:

$$\overline{d\ell} = \overline{dr} = \hat{\mathbf{a}}_r \, dr$$

Step 2: Evaluate the **dot product** $\mathbf{A}(\bar{r}_c) \cdot \overline{d\ell}$.

$$\begin{aligned}\mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} &= \left(z \ \hat{a}_x - x \ \hat{a}_y\right) \cdot \hat{a}_r \, dr \\ &= \left(z \ \hat{a}_x \cdot \hat{a}_r - x \ \hat{a}_y \cdot \hat{a}_r\right) dr \\ &= \left(z \ \sin\theta \cos\phi - x \ \sin\theta \sin\phi\right) dr\end{aligned}$$

Step 3: Transform all coordinates of the resulting **scalar** field to the **same** system as C.

The contour is a **spherical** contour. Recall that $z = r\cos\theta$ and $x = r\sin\theta\cos\phi$, therefore:

$$\begin{aligned}\mathbf{A}(\bar{r}_c) \cdot \overline{d\ell} &= \left(z \ \sin\theta \cos\phi - x \ \sin\theta \sin\phi\right) dr \\ &= \left(r\cos\theta \ \sin\theta\cos\phi - r\sin\theta\cos\phi \ \sin\theta\sin\phi\right) dr \\ &= r \ \sin\theta\cos\phi\left(\cos\theta - \sin\theta\sin\phi\right) dr\end{aligned}$$

Step 4: Evaluate the scalar field using the **two** coordinate **equalities** that describe contour C.

Recall that θ=60° and ϕ=45° at **every** point along the contour we are integrating over. Thus, functions of θ or ϕ are **constants** with respect to the integration! For example, $\cos\theta = \cos 45^\circ = 0.5$. Therefore:

$$\mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} = r\ \sin 60^\circ \cos 45^\circ \left(\cos 60^\circ - \sin 60^\circ \sin 45^\circ\right) dr$$

$$= r\sqrt{3/4}\sqrt{1/2}\left(1/2 - \sqrt{3/4}\sqrt{1/2}\right) dr$$

$$= r\sqrt{3/8}\left(\frac{\sqrt{2}-\sqrt{3}}{\sqrt{8}}\right) dr$$

$$= \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

Step 5: Determine the **limits of integration** from the **inequality** that describes contour *C (be careful of order!).*

We note the contour is described as:

$$0 \leq r \leq 4$$

and the contour C moves from $r = 0$ to $r = 4$. Thus, we integrate from 0 to 4 :

$$\int_C \mathbf{A}(\bar{r}_c)\cdot\overline{d\ell} = \int_0^4 \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

Note: if the contour ran from $r = 4$ to $r = 0$ the limits of integration would be **flipped**! I.E.,

$$\int_4^0 \left(\frac{\sqrt{6}-3}{8}\right) r\ dr$$

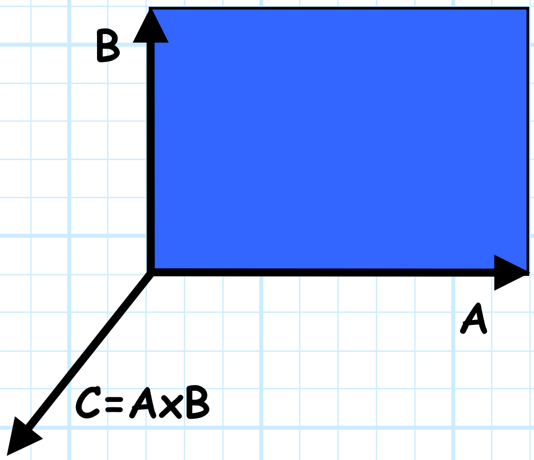
It is readily apparent that the line integral from $r = 0$ to $r = 4$ is the opposite (i.e., **negative**) of the integral from $r = 4$ to $r = 0$.

Step 6: Integrate the remaining function of **one** coordinate variable.

$$\begin{aligned}\int_C \mathbf{A}(\overline{r_c}) \cdot \overline{d\ell} &= \int_0^4 \left(\frac{\sqrt{6}-3}{8}\right) r\, dr \\ &= \left(\frac{\sqrt{6}-3}{8}\right) \int_0^4 r\, dr \\ &= \left(\frac{\sqrt{6}-3}{8}\right)\left(\frac{4^2}{2} - \frac{0^2}{2}\right) \\ &= \sqrt{6}-3\end{aligned}$$

Differential Surface Vectors

Consider a **rectangular surface**, oriented in some arbitrary direction:



We can describe this surface using **vectors**! One vector (say **A**), is a directed distance that denotes the **length** (i.e, magnitude) and **orientation** of one edge of the rectangle, while another directed distance (say **B**) denotes the length and orientation of the other edge.

Say we take the **cross-product** of these two vectors (**AxB=C**).

Q: *What does this vector **C** represent?*

A: Look at the **definition** of cross product!

$$\begin{aligned}\mathbf{C} &= \mathbf{A} \times \mathbf{B} \\ &= \hat{a}_n \left|\mathbf{A}\right| \left|\mathbf{B}\right| \sin\theta_{AB} \\ &= \hat{a}_n \left|\mathbf{A}\right| \left|\mathbf{B}\right|\end{aligned}$$

Note that:

$$\left|\mathbf{C}\right| = \left|\mathbf{A}\right| \left|\mathbf{B}\right|$$

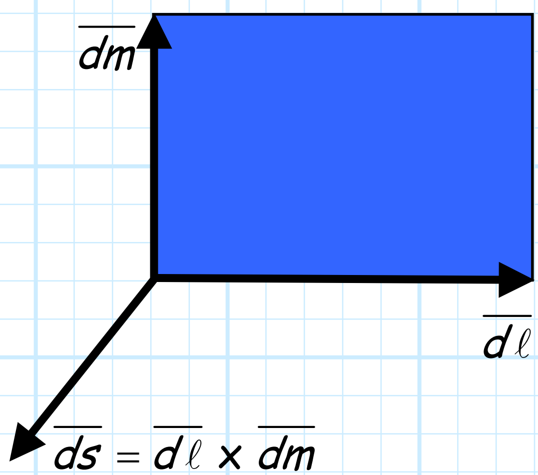
The magnitude of vector **C** is therefore product of the lengths of each directed distance—the **area of the rectangle**!

Likewise, $\mathbf{C} \cdot \mathbf{A} = 0$ and $\mathbf{C} \cdot \mathbf{B} = 0$, therefore vector **C** is orthogonal (i.e., "**normal**") to the **surface** of the rectangle.

As a result, vector **C** indicates **both** the **size** and the **orientation** of the rectangle.

The differential surface vector

For example, consider the **very small** rectangular surface resulting from two differential displacement vectors, say $\overline{d\ell}$ and $\overline{dm}$.



For example, consider the situation if $\overline{d\ell} = \overline{dx}$ and $\overline{dm} = \overline{dy}$:

$$\begin{aligned} \overline{ds} &= \overline{dx} \times \overline{dy} \\ &= \left(\hat{a}_x \times \hat{a}_y\right) dx\ dy \\ &= \hat{a}_z\ dx\ dy \end{aligned}$$

In other words the **differential** surface element has an **area** equal to the product $dx\, dy$, and a **normal vector** that points in the $\hat{a}_z$ direction.

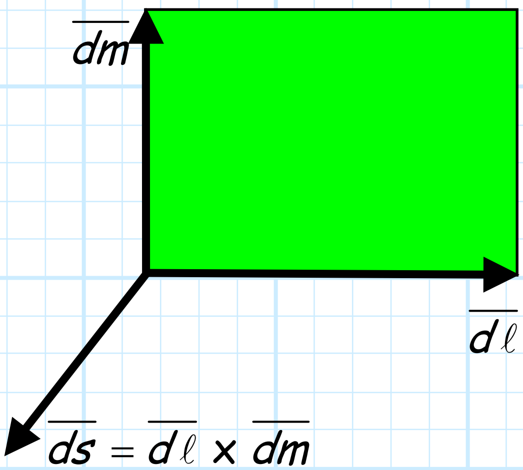
The differential surface vector $\overline{ds}$ specifies the size and orientation of a small (i.e., **differential**) **patch** of area, located on some arbitrary **surface** S.

We will use the differential surface vector in evaluating **surface integrals** of the type:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

The Differential Surface Vector for Coordinate Systems

Given that $\overline{ds} = \overline{d\ell} \times \overline{dm}$, we can determine the differential surface vectors for each of the **three** coordinate systems.



Cartesian

$$\overline{ds_x} = \overline{dy} \times \overline{dz} = \hat{a}_x \, dy \, dz$$
$$\overline{ds_y} = \overline{dz} \times \overline{dx} = \hat{a}_y \, dx \, dz$$
$$\overline{ds_z} = \overline{dx} \times \overline{dy} = \hat{a}_z \, dx \, dy$$

We shall find that these differential surface vectors define a small patch of area on the surface of **flat plane**.

Cylindrical

$$\overline{ds_\rho} = \overline{d\phi} \times \overline{dz} = \hat{a}_\rho \, \rho \, d\phi \; dz$$
$$\overline{ds_\phi} = \overline{dz} \times \overline{dp} = \hat{a}_\phi \, d\rho \; dz$$
$$\overline{ds_z} = \overline{d\rho} \times \overline{d\phi} = \hat{a}_z \, \rho \, d\rho \; d\phi$$

We shall find that $\overline{ds_\rho}$ describes a small patch of area on the surface of a **cylinder**, $\overline{ds_\phi}$ describes a small patch of area on the surface of a **half-plane**, and $\overline{ds_z}$ again describes a small patch of area on the surface of a flat **plane**.

Spherical

$$\overline{ds_r} = \overline{d\theta} \times \overline{d\phi} = \hat{a}_r \, r^2 \sin\theta \, d\theta \; d\phi$$
$$\overline{ds_\theta} = \overline{d\phi} \times \overline{dr} = \hat{a}_\theta \, r \sin\theta \, dr \; d\phi$$
$$\overline{ds_\phi} = \overline{dr} \times \overline{d\theta} = \hat{a}_\phi \, r \, dr \; d\theta$$

We shall find that $\overline{ds_r}$ describes a small patch of area on the surface of a **sphere**, $\overline{ds_\theta}$ describes a small patch of area on the surface of a **cone**, and $\overline{ds_\phi}$ again describes a small patch of area on the surface of a **half plane**.

The Surface Integral

An important type of vector integral that is often quite useful for solving physical problems is the **surface integral**:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

Some important things to note:

* The integrand is a **scalar** function.

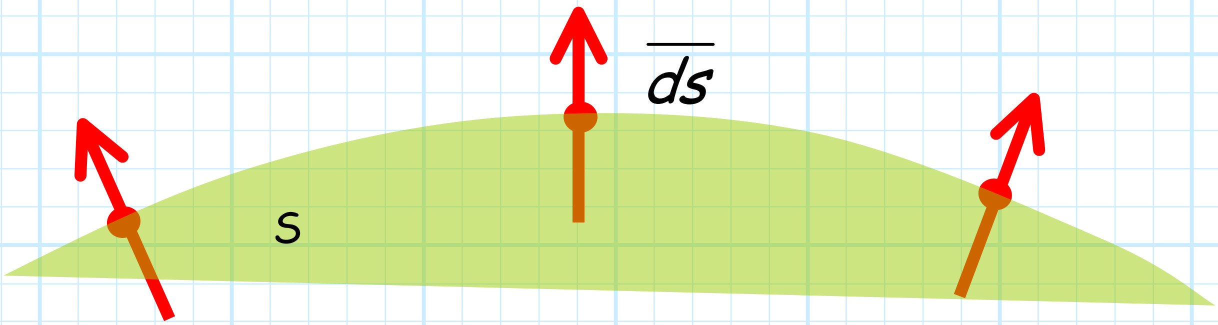
* The integration is over **two** dimensions.

* The **surface** S is an arbitrary two-dimensional surface in a three-dimensional space.

* The position vector $\bar{r}_s$ denotes only those points that lie on surface S. Therefore, the value of this integral **only** depends on the value of vector field $\mathbf{A}(\bar{r})$ at the points on this surface.

Q: *How are differential surface vector $\overline{ds}$ and surface S related?*

A: The differential vector $\overline{ds}$ describes a differential surface area at every point on S.



As a result, the differential surface vector $\overline{ds}$ is **normal** (i.e., orthogonal) to surface S at every point on S.

Q: *So what does the scalar integrand* $\mathbf{A}(\bar{r}_s)\cdot\overline{ds}$ *mean? What is it that we are actually integrating?*

A: Essentially, the surface integral integrates (i.e., "adds up") the values of a **scalar component** of vector field $\mathbf{A}(\bar{r})$ at **each and every point** on surface S. This scalar component of vector field $\mathbf{A}(\bar{r})$ is the projection of $\mathbf{A}(\bar{r}_s)$ onto a direction perpendicular (i.e., normal) to the surface S.

First, I must point out that the notation $\mathbf{A}(\bar{r}_s)$ is **non-standard**. Typically, the vector field in the surface integral is denoted simply as $\mathbf{A}(\bar{r})$. I use the notation $\mathbf{A}(\bar{r}_s)$ to emphasize that we are integrating the values of the vector field $\mathbf{A}(\bar{r})$ **only** at points that lie on surface S, and the points that lie on surface S are denoted by position vector $\bar{r}_s$.

In other words, the values of vector field $\mathbf{A}(\bar{r})$ at points that do **not** lie on the surface (which is just about all of them!) have **no effect** on the integration. The integral **only** depends on the value of the vector field as we move over surface S—we denote these values as $\mathbf{A}(\bar{r}_s)$.

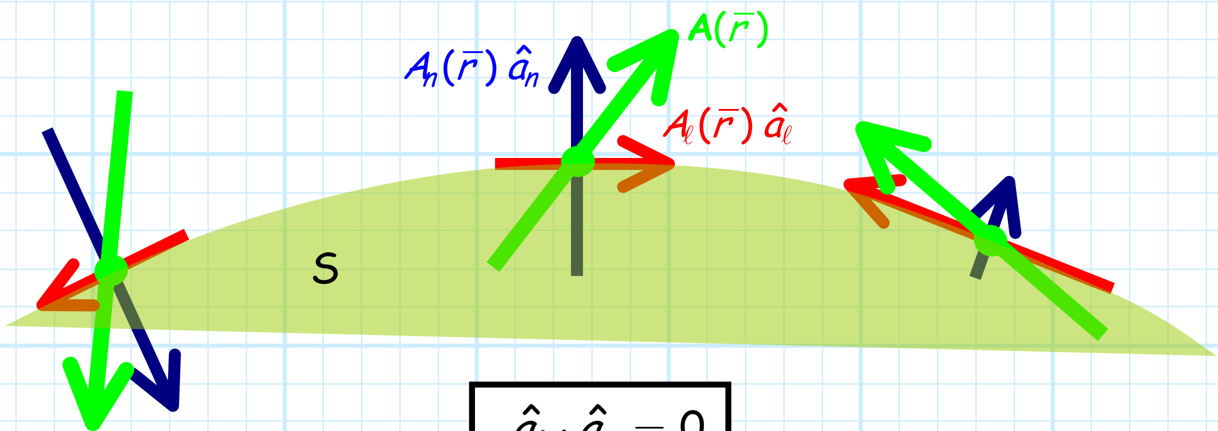
Moreover, the surface integral depends on **only one component** of $\mathbf{A}(\bar{r}_s)$!

Q: *On just what component of* $\mathbf{A}(\bar{r}_s)$ ***does*** *the integral depend?*

A: Look at the integrand $\mathbf{A}(\bar{r}_s)\cdot\overline{ds}$ --we see it involves the **dot product**! Thus, we find that the scalar integrand is simply the **scalar projection** of $\mathbf{A}(\bar{r}_s)$ onto the differential vector $\overline{ds}$. As a result, the integrand depends **only** the component of $\mathbf{A}(\bar{r}_s)$ that lies in the direction of $\overline{ds}$ --and $\overline{ds}$ **always** points in the direction orthogonal to surface S!

To help see this, first note that every vector $\mathbf{A}(\bar{r}_s)$ can be written in terms of a component tangential to the surface (i.e, $A_\ell(\bar{r}_s)\,\hat{a}_\ell$), and a component that is **normal** (i.e., orthogonal) to the surface (i.e, $A_n(\bar{r}_s)\,\hat{a}_n$):

$$\mathbf{A}(\bar{r}_s) = A_\ell(\bar{r}_s)\,\hat{a}_\ell + A_n(\bar{r}_s)\,\hat{a}_n$$



$$\hat{a}_\ell \cdot \hat{a}_n = 0$$

We note that the differential surface vector $\overline{ds}$ can be written in terms of its magnitude ($\left|\overline{ds}\right|$) and direction ($\hat{a}_n$) as:

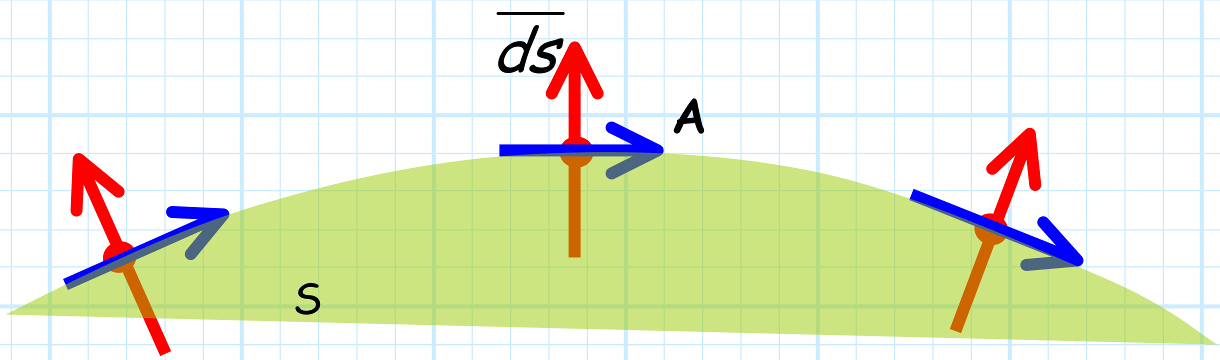
$$\overline{ds} = \hat{a}_n \left|\overline{ds}\right|$$

For example, for $\overline{ds_r} = \hat{a}_r\, r^2 \sin\theta\, d\theta d\phi$, we can say $\left|\overline{ds_r}\right| = r^2 \sin\theta\, d\theta\, d\phi$ and $\hat{a}_n = \hat{a}_r$.

As a result we can write:

$$
\begin{aligned}
\iint_S \mathbf{A}(\bar{r}) \cdot \overline{ds} &= \iint_S \left[A_\ell(\bar{r})\, \hat{a}_\ell + A_n(\bar{r})\, \hat{a}_n \right] \cdot \overline{ds} \\
&= \iint_S \left[A_\ell(\bar{r})\, \hat{a}_\ell + A_n(\bar{r})\, \hat{a}_n \right] \cdot\, \hat{a}_n \left|\overline{ds}\right| \\
&= \iint_S \left[A_\ell(\bar{r})\, \hat{a}_\ell \cdot \hat{a}_n + A_n(\bar{r})\, \hat{a}_n \cdot \hat{a}_n \right] \left|\overline{ds}\right| \\
&= \iint_S A_n(\bar{r}) \left|\overline{ds}\right|
\end{aligned}
$$

Note if vector field $\mathbf{A}(\bar{r})$ is **tangential** to the surface at every point, then the resulting surface integral will be **zero**.



Although S represents **any** surface, no matter how **complex** or **convoluted**, we will study only **basic** surfaces. In other words, $\overline{ds}$ will correspond to one of the differential surface vectors from Cartesian, cylindrical, or spherical coordinate systems.

The Surface S

In this class, we will limit ourselves to studying only those surfaces that are formed when we change the location of a point by varying **two** coordinate parameters. In other words, the other coordinate parameters will remain **fixed**.

Mathematically, therefore, a **surface** is described by:

1 equality (e.g., $x=2$ or $r=3$)

AND

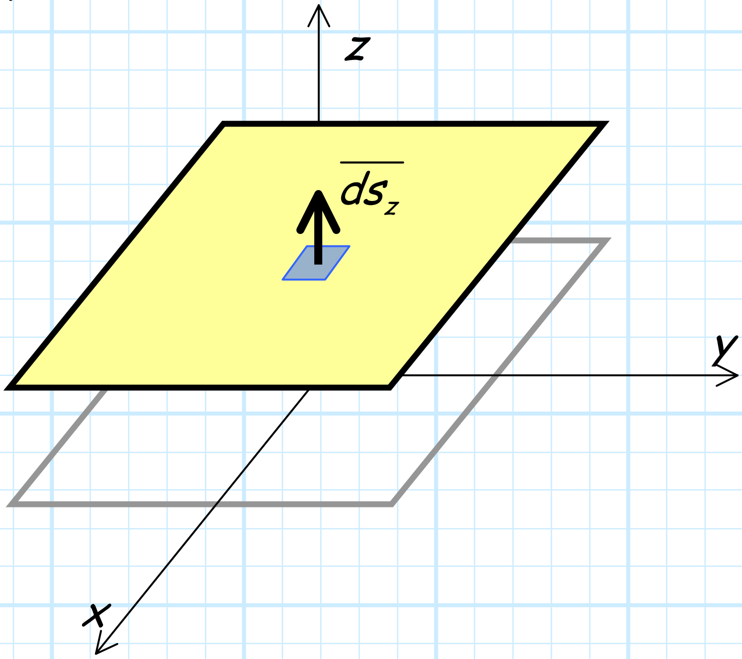
2 inequalities (e.g., $-1 < y < 5$ and $-2 < z < 7$, or $0 < \theta < \pi/2$ and $0 < \phi < \pi$)

Likewise, we will need to **explicitly** determine the **differential surface vector** $\overline{ds}$ for each contour.

We will be able to describe a surface for **each** of the coordinate values we have studied in this class!

Cartesian Coordinate Surfaces

The **single** equation $z = 3$ specifies **all** points P(x,y,z) with a coordinate value $z = 3$. These points form a plane that is **parallel** to the x-y plane.



* As we move across this plane, the coordinate values of x and y **will** vary. Thus, the size of this **rectangular** plane is defined by **two inequalities** --
  $c_{x1} \le x \le c_{x2}$ and $c_{y1} \le y \le c_{y2}$.

* Note the **differential surface vector** $\overline{ds_z}$ (or $-\overline{ds_z}$) is **orthogonal** to every point on this plane.

* Similarly, the equations $y = -2$ or $x = 6$ describe **planes** orthogonal to the x-z plane and the y-z plane, respectively. Likewise, the differential surface vectors $\overline{ds_y}$ and $\overline{ds_x}$ are orthogonal to each point on **these** planes.

Summarizing the **Cartesian surfaces**:

1. Flat **plane** parallel to the *y-z* plane.

$$x = c_x \qquad c_{y1} \leq y \leq c_{y2} \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_x} = \pm \hat{a}_x \, dy \, dz$$

2. Flat **plane** parallel to the *x-z* plane.

$$c_{x1} \leq x \leq c_{x2} \qquad y = c_y \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_y} = \pm \hat{a}_y \, dz \, dx$$

3. Flat **plane** parallel to the *x-y* plane.

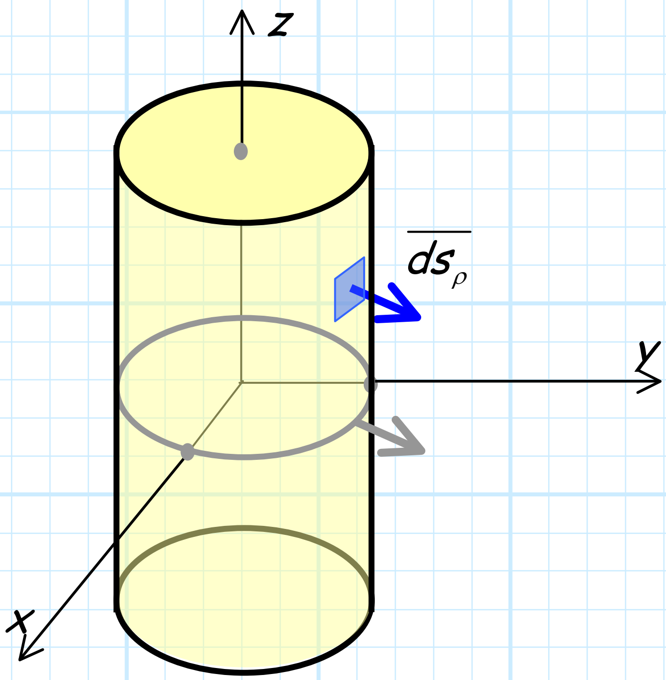
$$c_{x1} \leq x \leq c_{x2} \qquad c_{y1} \leq y \leq c_{y2} \qquad z = c_z$$

$$\overline{ds} = \pm \overline{ds_z} = \pm \hat{a}_z \, dy \, dx$$

Cylindrical Coordinate Surfaces

With **cylindrical** coordinates, we can define surfaces such as $\phi = 45^\circ$ or $\rho = 4$. These surfaces, however, are more **complex** than simply planes.

For example, the surface denoted by ρ=4 is formed by all points with coordinate ρ=4. In other words, this surface is formed by **all** points that are a distance of 4 units from the *z*-axis—a **cylinder** !

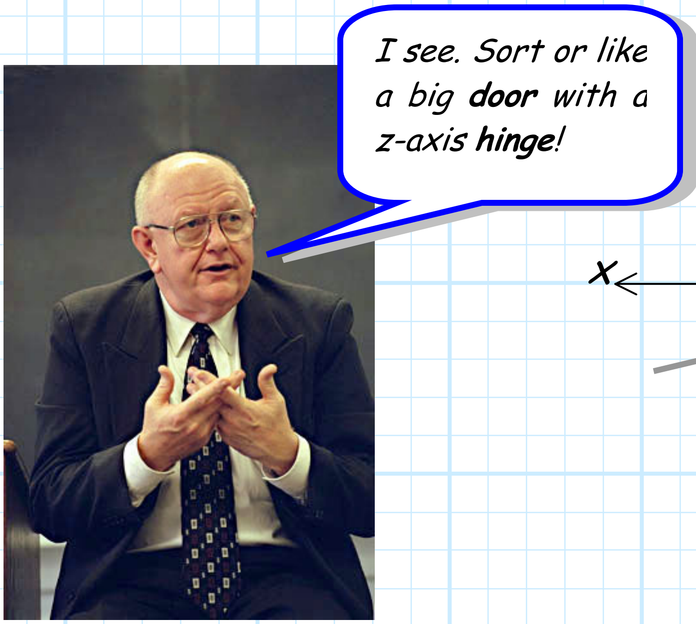


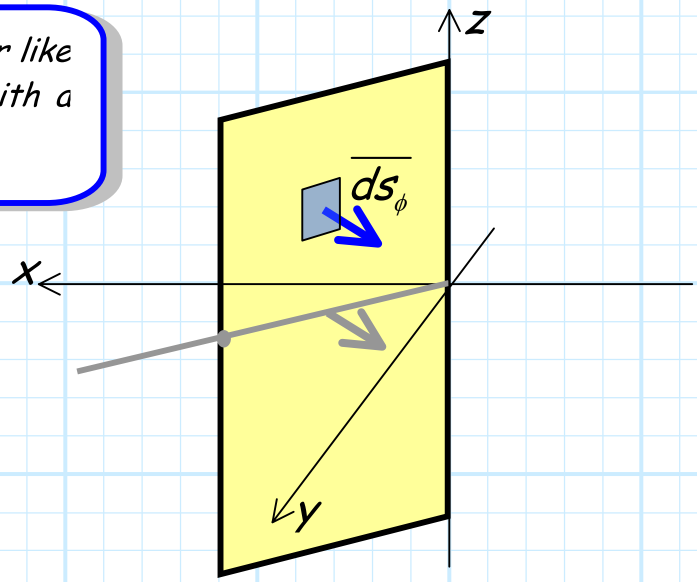
* As we move across this cylinder, the coordinate values of ϕ and *z* **will** vary. Thus, the size of this cylinder is defined by **two inequalities**--$c_{\phi 1} \leq \phi \leq c_{\phi 2}$ and $c_{z1} \leq z \leq c_{z2}$.

* Note a cylinder that **completely surrounds** the *z*-axis is described by the inequality $0 \leq \phi \leq 2\pi$. However, the cylinder does **not** have to be complete! For example, the inequality $0 \leq \phi \leq \pi$ defines a **half**-cylinder,

* We note the differential surface vector $\overline{ds}_\rho$ (or $-\overline{ds}_\rho$) is orthogonal to this surface at **all** points.

Another surface is defined by the equation $\phi = 45^\circ$. This surface is formed only from points with coordinate value $\phi = 45^\circ$. The surface is a **half-plane** that extends outward from the *z*-axis.



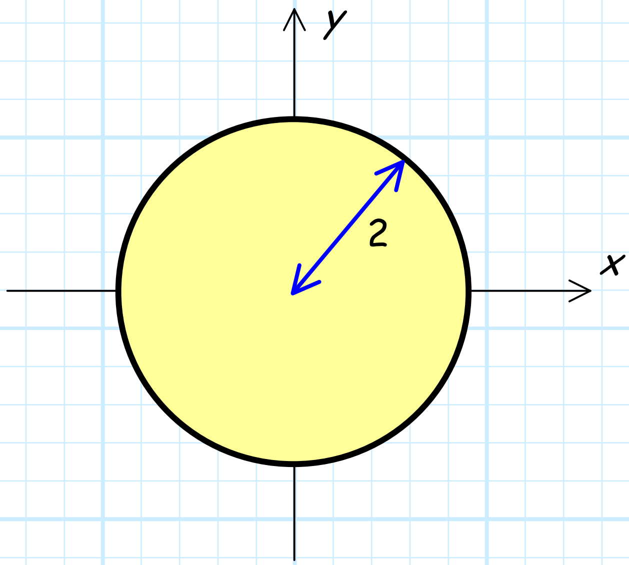


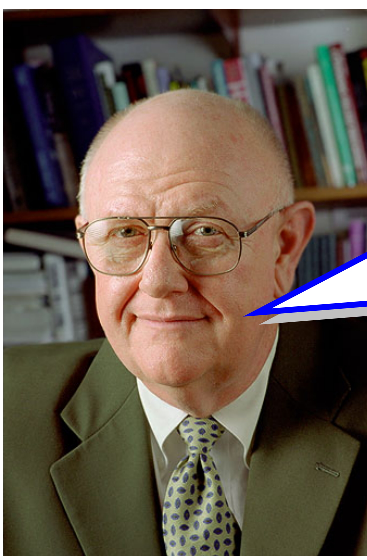
Note the differential surface vector $\overline{ds}_\phi$ is **orthogonal** to this surface at every point.

The **final cylindrical surface** that we will consider the type formed by the equality *z* = 2. We know that this forms a **flat plane** that is parallel to the *x*-*y* plane.

* Using the inequalities of **Cartesian** coordinates, this flat plane is rectangular in shape. However, using **cylindrical** coordinates inequalities, this plane will be shaped like a **ring** or a **disk**.

* For example, the surface $z = 0,\ 0 \le \rho \le 2,\ 0 \le \phi \le 2\pi$ describes a circular disk of radius 2, lying **on** the x-y plane, and centered at the **z**-axis:



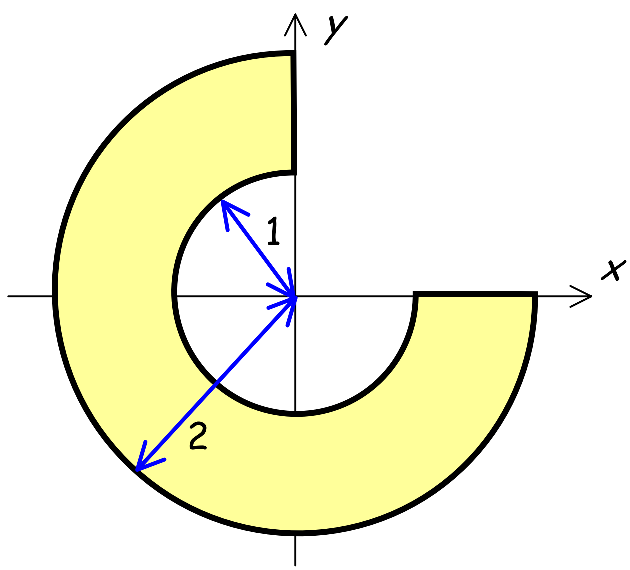


*Now let's see if **you've** been paying attention! Determine the two **inequalities** that define this flat surface.*

$$z = 0$$

$$\leq \rho \leq$$

$$\leq \phi \leq$$



Summarizing our **cylindrical surface** results:

1. Circular **cylinder** centered around the *z*-axis.

$$\rho = c_\rho \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\rho} = \pm \hat{a}_p \; \rho \, d\phi \, dz$$

2. "Vertical" **half-plane** extending from the *z*-axis.

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad \phi = c_\phi \qquad c_{z1} \leq z \leq c_{z2}$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi \; dz \, d\rho$$

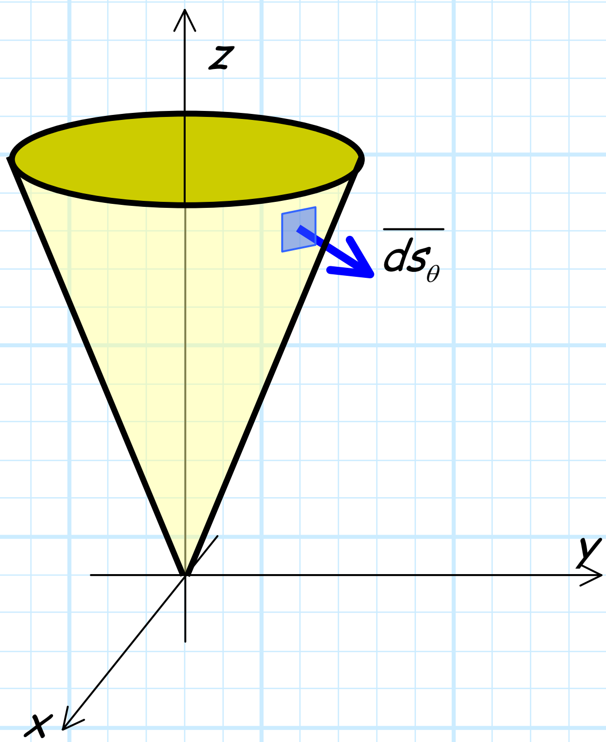
3. Flat **plane** parallel to the *x*-*y* plane.

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad z = c_z$$

$$\overline{ds} = \pm \overline{ds_z} = \pm \hat{a}_z \; \rho \, d\phi \, d\rho$$

Spherical Coordinate Surfaces

The surface defined by $\theta = 30^\circ$ is formed only from points with coordinate $\theta = 30^\circ$. This surface is a **cone**! The apex of the cone is centered at the origin, and its axis of rotation is the z-axis.



* Note that the differential surface vector $\overline{ds_\theta}$ is **normal** to this surface at every point.

* Just like a cylinder, a **complete** cone is defined by the inequality $0 \le \phi \le 2\pi$. Alternatively, for example, the equation $\pi \le \phi \le 3\pi/2$ defines a **quarter** cone.

Say instead our equality equation is $r = 3$. This defines a surface formed from all points a distance of 3 units from the origin—a **sphere** of radius 3 !

* This sphere is **centered** at the origin.

* The differential surface vector $\overline{ds_r}$ is normal to this sphere at all points on the surface.

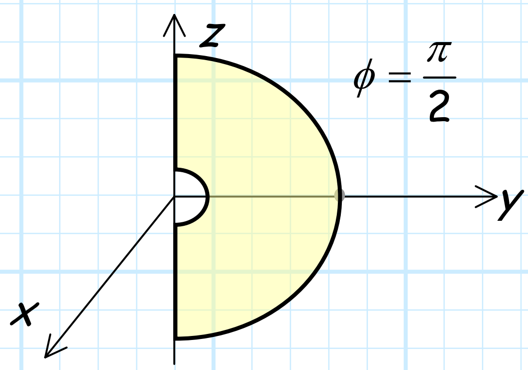
* If we wish to define a **complete** sphere, our inequalities must be:

$$0 \leq \theta < \pi \qquad \text{and} \qquad 0 \leq \phi < 2\pi$$

otherwise, we will be defining some **subsection** of a spherical surface (e.g., the "Northern Hemisphere".).

Finally, we know that the equation $\phi = 45^\circ$ defines a vertical **half-plane**, extending from the z-axis.

However, using **spherical** inequalities, this vertical plane will be in the shape of a **semi-circle** (or some section thereof), as opposed to rectangular (with cylindrical inequalities).



Summarizing the **spherical surfaces**:

1. **Sphere** centered at the origin.

$$r = c_r \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_r} = \pm \hat{a}_r \, r^2 \sin\theta \, d\theta \, d\phi$$

2. A **cone** with apex at the origin and aligned with the *z*-axis.

$$c_{r1} \leq r \leq c_{r2} \qquad \theta = c_{\theta} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

$$\overline{ds} = \pm \overline{ds_\theta} = \pm \hat{a}_\theta \, r \sin\theta \, d\phi \, dr$$

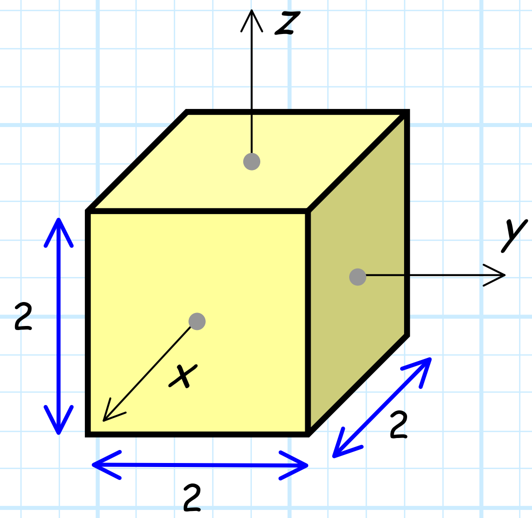
3. "Vertical" **half-plane** extending **from** the *z*-axis.

$$c_{r1} \leq r \leq c_{r2} \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad \phi = c_{\phi}$$

$$\overline{ds} = \pm \overline{ds_\phi} = \pm \hat{a}_\phi \, r \, dr \, d\theta$$

Integrals with Complex Surfaces

Similar to contours, we can form complex surfaces by combining any of the **seven** simple surfaces that can easily be formed with Cartesian, cylindrical or spherical coordinates. For example, we can define **6 planes** to form the surface of a **cube** centered at the origin:



The cube surface S is thus described as the sum of the **six** sides:

$$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$$

Therefore, a surface integration over S can be evaluated as:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot\overline{ds} = \iint_{S_1} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_3} \mathbf{A}(\bar{r}_s)\cdot\overline{ds}$$
$$+ \iint_{S_4} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_5} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_6} \mathbf{A}(\bar{r}_s)\cdot\overline{ds}$$

This is a great example for considering the **direction** of differential surface vector $\overline{ds}$.

Recall there are **two** differential surface vectors that are orthogonal to every surface: the first is simply the **opposite** of the second.

For example, if we were performing a surface integration over the top surface of this cube (i.e., z=1 plane) , we would **typically** use $\overline{ds} = \overline{ds}_z = \hat{a}_z \, dx \, dy$.

However, we could **also** use the differential surface vector $\overline{ds} = -\overline{ds}_z = -\hat{a}_z \, dx \, dy$!

Q: *How would the results of the two integrations* ***differ?***

A: By a factor of **−1** !!

We find that a surface integration using $\overline{ds}$ is related to the surface integration using $-\overline{ds}$ as:

$$\iint_S \mathbf{A}(\overline{r}_s) \cdot (-\overline{ds}) = -\iint_S \mathbf{A}(\overline{r}_s) \cdot \overline{ds}$$

The surface of a cube is an example of a **closed surface.** A closed surface is a surface that **completely surrounds** some volume. You cannot get from **one side** of a closed surface to the **other** side without **passing through** the surface.

In other words, if your **beverage** is surrounded by a closed surface, better go get your **can opener**!

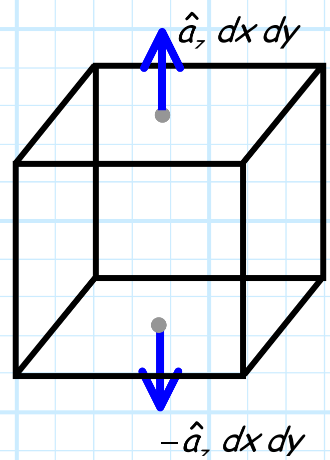
In electromagnetics, we **often** define $\overline{ds}$ as the direction **pointing outward** from a **closed** surface.

So, for example, the differential surface vector for the **top** surface (z=1) would be:

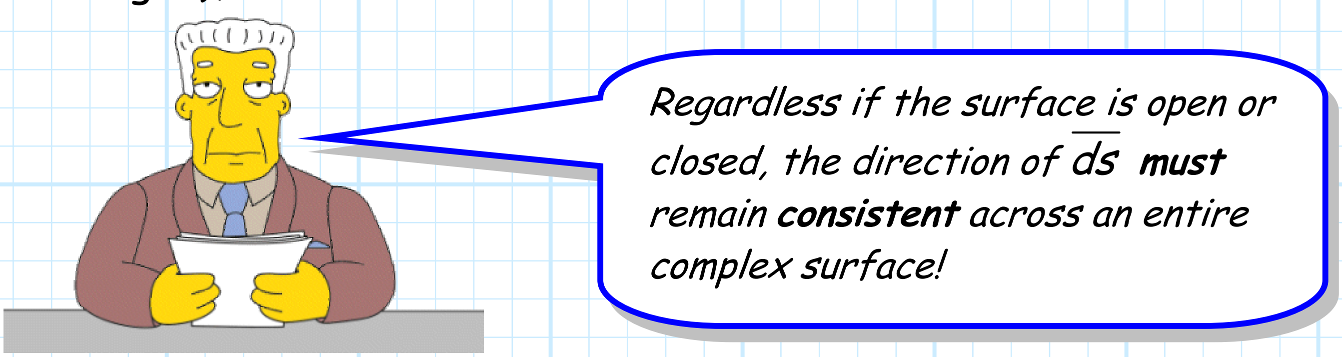
$$\overline{ds} = \overline{ds_z} = \hat{a}_z \, dx \, dy ,$$

while on the **bottom** (z=-1) we would use :

$$\overline{ds} = -\overline{ds_z} = -\hat{a}_z \, dx \, dy$$



Similarly, we would use differential line vectors of **opposite** directions for each of the pair of side surfaces (left and right), as well as for the front and back surfaces.



Steps for Analyzing Surface Integrals

We wish to **evaluate** an integral of the form:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

To successfully accomplish this, simply follow these steps:

Step 1: Determine the 1 equality, 2 inequalities, and $\overline{ds}$ for the surface S (be careful of direction!).

Step 2: **Evaluate** the dot product $\mathbf{A}(\bar{r}_s) \cdot \overline{ds}$.

Step 3: Write the resulting scalar field using the **same** coordinate system as surface S.

Step 4: Evaluate the scalar field using the coordinate **equality** that described surface S.

Step 5: Determine the **limits of integration** from the **inequalities** that describe surface S.

Step 6: Integrate the remaining function of **two** coordinate variables.

Example: The Surface Integral

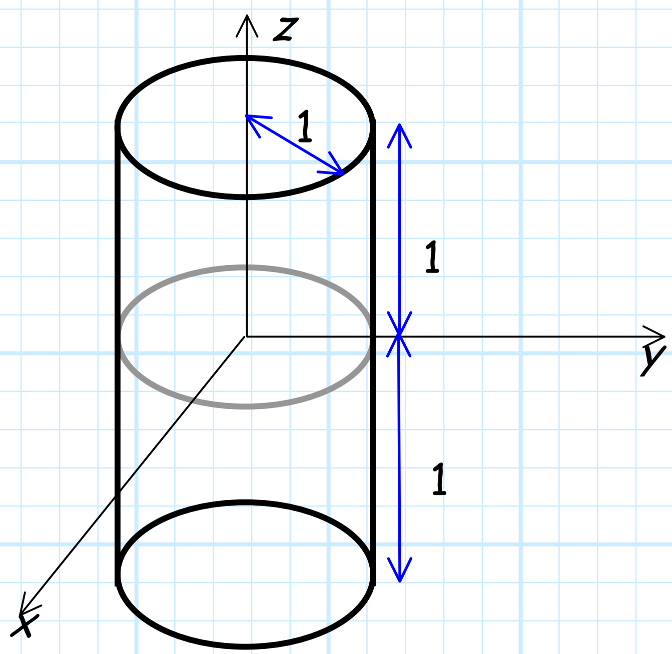
Consider the vector field:

$$\mathbf{A}(\bar{r}) = x\ \hat{a}_x$$

Say we wish to **evaluate** the surface integral:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

where S is a **cylinder** whose axis is aligned with the z-axis and is centered at the origin. This cylinder has a **radius** of 1 unit, and extends 1 unit below the x-y plane and one unit above the x-y plane. In other words, the cylinder has a **height** of 2 units.



This is a **complex**, **closed** surface. We will define the **top** of the cylinder as surface S_1, the **side** as S_2, and the **bottom** as S_3. The surface integral will therefore be evaluated as:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot\overline{ds} = \iint_{S_1} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_1} + \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_2} + \iint_{S_3} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_3}$$

Step 1: Determine $\overline{ds}$ for the surface S.

Let's define $\overline{ds}$ as pointing in the direction outward from the closed surface.

S_1 is a **flat plane** parallel to the *x-y* plane, defined as:

$$0 \le \rho \le 1 \quad 0 \le \phi \le 2\pi \quad z = 1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_1} = \overline{ds_z} = \hat{a}_z \; \rho \, d\rho \, d\phi$$

S_2 is a **circular cylinder** centered on the *z*- axis, defined as:

$$\rho = 1 \quad 0 \le \phi \le 2\pi \quad -1 \le z \le 1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_2} = \overline{ds_\rho} = \hat{a}_\rho \; \rho \, dz \, d\phi$$

S_3 is a **flat plane** parallel to the *x-y* plane, defined as:

$$0 \le \rho \le 1 \quad 0 \le \phi \le 2\pi \quad z = -1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_3} = -\overline{ds_z} = -\hat{a}_z \; p\, dp\, d\phi$$

Step 2: Evaluate the dot product $\mathbf{A}(\bar{r}_s) \cdot \overline{ds}$.

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_1} &= x\; \hat{a}_x \cdot \hat{a}_z \;\rho\, d\rho\, d\phi \\ &= x\,(0)\,\rho\, d\rho\, d\phi \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} &= x\; \hat{a}_x \cdot \hat{a}_p \;\rho\, dz\, d\phi \\ &= x\,(\cos\phi)\,\rho\, dz\, d\phi \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_3} &= -x\; \hat{a}_x \cdot \hat{a}_z \;\rho\, d\rho\, d\phi \\ &= -x\,(0)\,\rho\, d\rho\, d\phi \\ &= 0 \end{aligned}$$

Look! Vector field $A(\bar{r})$ is **tangential** to surface S_1 and S_3 for all points on surface S_1 and S_3! Therefore:

$$\begin{aligned} \iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds} &= \iint_{S_1} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_1} + \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} + \iint_{S_3} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_3} \\ &= 0 + \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} + 0 \\ &= \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} \end{aligned}$$

Step 3: Write the resulting scalar field using the same coordinate system as $\overline{ds}$.

The differential vector $\overline{ds_\rho}$ is expressed in **cylindrical** coordinates, therefore we must write the **scalar** integrand using cylindrical coordinates.

We know that:

$$x = \rho \cos\phi$$

Therefore:

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} &= x(\cos\phi)\rho \, dz \, d\phi \\ &= \rho\cos\phi(\cos\phi)\rho \, dz \, d\phi \\ &= \rho^2\cos^2\phi \; dz \, d\phi \end{aligned}$$

Step 4: Evaluate the scalar field using the coordinate **equality** that described surface S.

Every point on S_2 has the coordinate value $\rho = 1$. Therefore:

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} &= \rho^2\cos^2\phi \; dz \, d\phi \\ &= 1^2\cos^2\phi \; dz \, d\phi \\ &= \cos^2\phi \; dz \, d\phi \end{aligned}$$

Step 5: Determine the **limits of integration** from the **inequalities** that describe surface S.

For S_2 we know that $0 \le \phi \le 2\pi \quad -1 \le z \le 1$.

Therefore:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot \overline{ds} = \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot \overline{ds_2} = \int_0^{2\pi}\int_{-1}^{1} \cos^2\phi \; dz\, d\phi$$

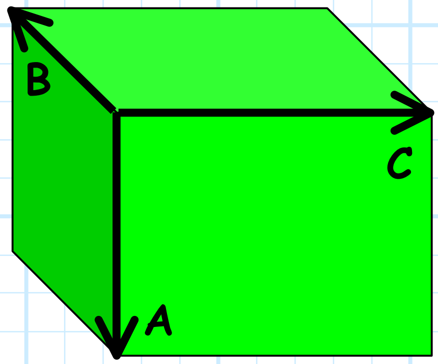
Step 6: Integrate the remaining function of **two** coordinate variables.

Using **all** the results determined above, the surface integral becomes:

$$\begin{aligned}\iint_S \mathbf{A}(\bar{r}_s)\cdot \overline{ds} &= \int_0^{2\pi}\int_{-1}^{1} \cos^2\phi \; dz\, d\phi \\ &= \int_0^{2\pi} \cos^2\phi \; d\phi \int_{-1}^{1} dz \\ &= (\pi - 0)(1-(-1)) \\ &= \mathbf{2\pi}\end{aligned}$$

The Differential Volume Element

Consider a **rectangular cube**, whose **three** sides can be defined by **three** directed distances, say **A**, **B**, and **C**.



It is evident that the lengths of each side of the rectangular cube are $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{C}|$, such that the **volume** of this rectangular cube can be expressed as:

$$V = |\mathbf{A}||\mathbf{B}||\mathbf{C}|$$

Consider now what happens if we take the **triple product** of these three vectors:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \cdot \hat{a}_n |\mathbf{B}||\mathbf{C}| \sin\theta_{BC}$$

However, we note that $\sin\theta_{BC} = \sin 90^\circ = 1.0$, and that $\hat{a}_n = \hat{a}_A$ (i.e., vector $\mathbf{B} \times \mathbf{C}$ points in the same direction as vector **A**!).

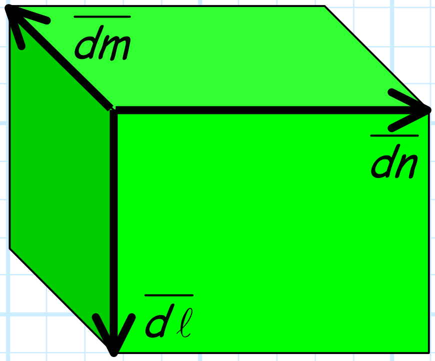
Using the fact that $\mathbf{A} = |\mathbf{A}|\,\hat{a}_A$, we then find the result:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} &= \mathbf{A}\cdot\hat{a}_n\,|\mathbf{B}||\mathbf{C}|\sin\theta_{BC}\\ &= \mathbf{A}\cdot\hat{a}_A\,|\mathbf{B}||\mathbf{C}|\\ &= |\mathbf{A}|\,\hat{a}_A\cdot\hat{a}_A\,|\mathbf{B}||\mathbf{C}|\\ &= |\mathbf{A}||\mathbf{B}||\mathbf{C}|\end{aligned}$$

Look what this means, the **volume** of a cube can be expressed in terms of the **triple product**!

$$V = \mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = |\mathbf{A}||\mathbf{B}||\mathbf{C}|$$

Consider now a rectangular volume formed by three orthogonal **line vectors** (e.g., $\overline{dx}$, $\overline{dy}$, $\overline{dz}$ or $\overline{d\rho}$, $\overline{d\phi}$, $\overline{dz}$).



The result is a differential volume, given as:

$$dv = \overline{d\ell}\cdot\overline{dm}\times\overline{dn}$$

For example, for the **Cartesian** coordinate system:

$$
\begin{aligned}
dv &= \overline{dx} \cdot \overline{dy} \times \overline{dz} \\
&= dx\, dy\, dz
\end{aligned}
$$

and for the **cylindrical** coordinate system:

$$
\begin{aligned}
dv &= \overline{d\rho} \cdot \overline{\mathrm{d}\phi} \times \overline{dz} \\
&= \rho\, d\rho\, d\phi\, dz
\end{aligned}
$$

and also for the **spherical** coordinate system:

$$
\begin{aligned}
dv &= \overline{dr} \cdot \overline{\mathrm{d}\theta} \times \overline{\mathrm{d}\phi} \\
&= r^2 \sin\theta\, dr\; d\phi\, d\theta
\end{aligned}
$$

The Volume V

As we might expect from out knowledge about how to specify a **point** P (3 equalities), a **contour** C (2 equalities and 1 inequality), and a **surface** S (1 equality and 2 inequalities), a **volume** V is defined by **3 inequalities**.

Cartesian

The inequalities:

$$c_{x1} \leq x \leq c_{x2} \qquad c_{y1} \leq y \leq c_{y2} \qquad c_{z1} \leq z \leq c_{z2}$$

define a **rectangular volume**, whose sides are parallel to the *x-y*, *y-z*, and *x-z* planes.

The differential volume dv used for constructing this Cartesian volume is:

$$dv = dx\, dy\, dz$$

Cylindrical

The inequalities:

$$c_{\rho 1} \leq \rho \leq c_{\rho 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2} \qquad c_{z1} \leq z \leq c_{z2}$$

defines a **cylinder**, or some **subsection** thereof (e.g. a **tube**!).

The differential volume dv is used for constructing this cylindrical volume is:

$$dv = \rho \, d\rho \, d\phi \, dz$$

Spherical

The equations:

$$c_{r1} \leq r \leq c_{r2} \qquad c_{\theta 1} \leq \theta \leq c_{\theta 2} \qquad c_{\phi 1} \leq \phi \leq c_{\phi 2}$$

defines a **sphere**, or some subsection thereof (e.g., an "**orange slice**" !).

The differential volume dv used for constructing this spherical volume is:

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

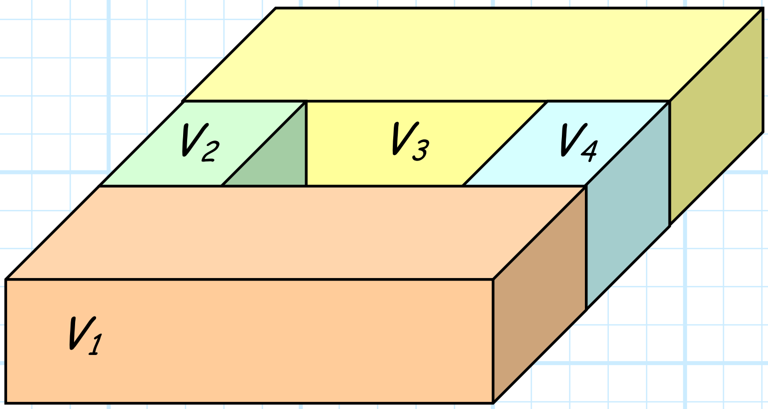
* Note that the three inequalities become **the limits of integration** for a volume integral. For example, integrating over a spherical volume would result in an integral of the form:

$$\iiint_V g(\bar{r}) \, dv = \int_{c_{\phi 1}}^{c_{\phi 2}} \int_{c_{\theta 1}}^{c_{\theta 2}} \int_{c_{r1}}^{c_{r2}} g(\bar{r}) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

For this example, **if** the scalar field $g(\bar{r})$ is **not** expressed in terms of **spherical** coordinates, it must first be **transformed** before the integral can be explicitly **evaluated**.

* Note also that we can construct **complex volumes** by combining the simple volumes discussed here.

$$V = V_1 + V_2 + V_3 + V_4$$



Example: The Volume Integral

Let's evaluate the **volume** integral:

$$\iiint_V g(\bar{r})\, dv$$

where $g(\bar{r}) = 1$ and the volume V is a **sphere** with radius R.

In other words, the volume V is described as:

$$0 \leq r \leq R$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$

And thus we use for the **differential** volume *dv:*

$$dv = \overline{dr} \cdot \overline{d\theta} \times \overline{d\phi} = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Therefore:

$$\iiint_V g(\bar{r})\,dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{R} r^2 dr$$

$$= 2\pi(2)\frac{R^3}{3}$$

$$= \frac{4}{3}\pi R^3$$

Hey look! The answer is the **volume** (e.g., in m^3) of a **sphere**!

Now, this result provided the numeric volume of V **only** because $g(\bar{r}) = 1$. We find that the total volume of **any** space V can be determined this way:

$$\text{Volume of } V = \iiint_V (1)\, dv$$

Typically though, we find that $g(\bar{r}) \neq 1$, and thus the volume integral does **not** provide the numeric volume of space V.

Q: *So what's the volume integral even good for?*

A: Generally speaking, the scalar function $g(\bar{r})$ will be a density function, with units of **things/unit volume**. Integrating $g(\bar{r})$ with the volume integral provides us the **number of things** within the space V!

For example, let's say $g(\bar{r})$ describes the **density** of a big **swarm of insects**, using units of ***insects/m³*** (i.e., insects are the **things**). Note that $g(\bar{r})$ must indeed a **function** of position, as the density of insects changes at different locations throughout the swarm.

Now say we want to know the total number of insects within the swarm, which occupies some space *V*. We can determine this by simply applying the volume integral!

$$\text{number of insects in swarm } = \iiint_V g(\bar{r})\, dv$$

where space *V* completely encloses the insect swarm.